

THE VERSAL DEFORMATION SPACE OF A REFLEXIVE MODULE ON A RATIONAL CONE

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ABSTRACT. By an approach based on results of A. Ishii, we describe the versal deformation space of any reflexive module on the cone over the rational normal curve of degree m . To each component a resolution is given as the total space of a vector bundle on a Grassmannian. The vector bundle is a sum of copies of the cotangent bundle, the canonical sub-bundle, the dual of the canonical quotient bundle, and the trivial line bundle. Via an embedding in a trivial bundle, we obtain the components by projection. In particular we give equations for the minimal stratum in the Chern class filtration of the versal deformation space. We obtain a combinatorial description of the local deformation relation and a classification of the components. In particular we give a formula for the number of components.

1. INTRODUCTION

The versal deformation space is in general highly singular and difficult to describe. Some explicit results have been given, e.g. for surface singularities [7, 23, 1], and for torsion free sheaves on singular curves [21, 22]. The aim of this article is to describe the versal deformation space of any (not necessarily indecomposable) reflexive module M on the cone over the rational normal curve of degree m .

Assuming X is a rational surface singularity, A. Ishii proves in [16, 4.9] an interesting theorem giving a filtration of the versal deformation space R for deformations of a reflexive module M on X , which, in the case X is a rational double point, is the stratification with respect to isomorphism classes of modules. More precisely; let $\pi : \tilde{X} \rightarrow X$ be a minimal resolution, then a reflexive module M on X corresponds to a full sheaf \tilde{M} on \tilde{X} . For each $d \in \text{Pic } \tilde{X}$, Ishii defines a functor of families (parametrised by arbitrary schemes over R_{red}) of semi-full sheaves \mathcal{E} on \tilde{X} with an isomorphism. The functor is represented by a regular scheme F^d which is projective over R_{red} . As d varies, a finite stratification $\coprod S^d$ of R_{red} is obtained such that if the fibre of the versal family at $t \in R$ is the reflexive module N , then $t \in S^d$ if and only if the full sheaf \tilde{N} has Chern class d . Moreover; S^d is regular for all d . In particular; each component in the reduced versal deformation space is given as the closure of an S^d . By the McKay correspondence this gives the stratification by isomorphism classes if X is a rational double point. In the latter case Ishii also gives an explicit example of an F^d ; assume $c_1(\tilde{M})$ is minus the fundamental cycle, then F^0 is the minimal resolution of $X \cong R_{\text{red}}$, [16, 5.3]. If X is a rational double point, Ishii describes the closure of the minimal stratum in terms of resolutions, in [16, 5.6], and in particular obtains the local deformation relation [16, 5.5].

In the case X is the cone over the rational normal curve of degree m , there are m isomorphism classes of rank one reflexive modules and any reflexive module M is a direct sum of these. We find F^d for all M and all d . In Theorem 1 an intrinsic

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description is given: F^d (as k -scheme) is the total space of a vector bundle of relative extensions $\mathcal{E}xt_{\tilde{X} \times A/A}^1(\mathcal{E}_A, \mathcal{E}_A)$ on a Grassmannian A , where (A, \mathcal{E}_A) represents a functor of embeddings of semi-full sheaves \mathcal{E} with $c_1(\mathcal{E}) = d$, see Proposition 1.

In Theorem 2 we calculate $\mathcal{E}xt_{\tilde{X} \times A/A}^1(\mathcal{E}_A, \mathcal{E}_A)$ as a sum of copies of the cotangent bundle, the canonical sub-bundle, the dual of the canonical quotient bundle, and the trivial line bundle on A . The number of copies is given by the dimension of certain cohomology groups associated to a sub-sheaf of \tilde{M} . Remark that this strengthens and generalises the description of the “generic” minimal stratum in Ishii’s [16, 5.6iii]. Theorem 2 also gives an embedding of the vector bundle in the trivial vector bundle $\text{Ext}_X^1(M, M) \times A$ and the map to R is obtained as the composition of the embedding with the projection to $\text{Ext}_X^1(M, M)$. From the equations in Corollary 1 defining the embedding, an explicit expression for the image R^d of F^d in R for all the minimal strata is obtained in Corollary 2; R^d is the cone over a Segre embedding times an incidence variety times an affine space intersected with certain hyperplanes and quadratic hypersurfaces. In Corollary 3 we give an ideal I_d of minors which gives T^d by blowing up R^d . It gives T^d as the strict transform in resolutions of rank singularities and we observe how the Chern class filtration of the versal deformation space is related to the rank filtration.

From Theorem 1 a combinatorial description of the local deformation relation is obtained in Lemma 3. In contrast to the rational double point case, there are many non-trivial Chern class preserving deformations, they give smooth strata in R_{red} . In Theorem 3 the components of the reduced versal deformation space are classified and a formula for the number of components is given. The components correspond to the geometrically rigid modules, and they are listed in Corollary 4. Further observations concerning the local deformation relation are given in Corollaries 4–8. Three elementary examples are found in the final section.

The study of reflexive modules on rational surface singularities may be traced back to the 1960’s. D. Mumford in characteristic zero [18] and J. Lipman in characteristic $p > 0$ [17] proved that a surface singularity X is rational [3] if and only if X has finitely many isomorphism classes of rank one reflexive modules. Later J. Herzog [14], H. Esnault [10] and M. Auslander [6] proved that a rational surface singularity is a quotient singularity if and only if there is a finite number of indecomposable reflexive modules on X . As shown in [11, 2], the intersection of the Chern class of the full sheaf with the exceptional divisor, sets up a correspondence between the set of isomorphism classes of non-trivial indecomposable reflexive modules and the components of the exceptional in the case X is a rational double point. This is the McKay correspondence. The Chern character (i.e. rank and Chern class) does not determine the corresponding reflexive module for general quotient singularities, cf. [10]. In [25] J. Wunram determined the full sheaves for cyclic quotient singularities, and following [10] he gave in [26] a cohomological criterion on a full sheaf such that a generalised McKay correspondence may be set up for the corresponding sub-class of indecomposable reflexive modules.

2. PRELIMINARIES

In this section we introduce notation which is fixed and cite standard results which will be used freely throughout the article. Let X be a *surface singularity*, i.e. $X = \text{Spec } \mathcal{O}_X$ where \mathcal{O}_X is the Henselisation of a local, normal, essentially finitely generated k -algebra of dimension 2 over an algebraically closed field k of any characteristic. There exists a minimal resolution $\pi : \tilde{X} \rightarrow X$ of the singularity in all characteristics, [17, 4.1], and X is a *rational surface singularity* if $R^1\pi_*\mathcal{O}_{\tilde{X}} = 0$, [3]. Remark that $H^i(\tilde{X}, \mathcal{F}) = 0$ for any coherent sheaf \mathcal{F} and $i \geq 2$ by the

Theorem on Formal Functions, cf. [13, 11.1]. In this article X will be the affine cone over the m -uple embedding of \mathbb{P}_k^1 in \mathbb{P}_k^m , i.e. $\mathcal{O}_X = k[u^m, u^{m-1}v, \dots, v^m]^h$ where “h” denotes Henselisation. The exceptional divisor $C = \tilde{X} \times_X \text{Spec } k \subseteq \tilde{X}$ is therefore isomorphic to \mathbb{P}_k^1 . There is an intersection theory on \tilde{X} , see [18, 3, 17], and $C \sim -mD$ where D is any curve intersecting C transversally in one point, in particular $C^2 = -m$. Moreover; by [17] we have $\text{Pic } \tilde{X} \cong \mathbb{Z}$ generated by D .

A finitely generated \mathcal{O}_X -module M is called *reflexive* if the canonical map to its double \mathcal{O}_X -dual, $M \rightarrow M^{\vee\vee}$, is an isomorphism. Since X is 2-dimensional and normal, a reflexive module is the same as a maximal Cohen-Macaulay module. In particular; M restricted to the regular locus $U \subseteq X$ is locally free. Let $\tilde{M} = \pi^*M/\text{torsion}$ and more generally, if M_S is an S -flat family of reflexive modules on X for a k -scheme S , then \tilde{M}_S is the image of the canonical map from $\pi_S^*M_S$ to its double $\mathcal{O}_{\tilde{X} \times S}$ -dual where $\pi_S : \tilde{X} \times S \rightarrow X \times S$ is the pullback of π . Following [10], \tilde{M} is called a *full* sheaf, and $M = H^0(X, \pi_*\tilde{M})$. Moreover; a sheaf \mathcal{E} on \tilde{X} is shown to be full if and only if \mathcal{E} is locally free, generated by global sections and $R^1\pi_*\mathcal{E}^\omega = 0$, where $\mathcal{E}^\omega := \text{Hom}_{\tilde{X}}(\mathcal{E}, \omega_{\tilde{X}})$. In particular $M = H^0(X, \pi_*\mathcal{E})$ is a reflexive \mathcal{O}_X -module with $\tilde{M} = \mathcal{E}$ since they are generated by the same global sections. For our particular X we have rank one reflexive \mathcal{O}_X -modules $M_i = (u^i, u^{i-1}v, u^{i-2}v^2, \dots, v^i)$ for $0 \leq i \leq m-1$ with $\tilde{M}_i \cong \mathcal{O}_{\tilde{X}}(iD)$. Since the group $H = \text{coker}(\mathbb{Z} \xrightarrow{\cdot C} \text{Pic } \tilde{X}) \cong \mathbb{Z}/m\mathbb{Z}$ by [17] classifies the rank one reflexive modules, the following lemma implies that the M_i are the only indecomposables.

Lemma 1. *If M is a reflexive module on X , then M is isomorphic to a direct sum of rank one reflexive modules.*

Proof. It is sufficient to show that \tilde{M} is a direct sum of line bundles. More generally we show that a vector bundle \mathcal{E} on \tilde{X} is isomorphic to a direct sum of line bundles. Let i be maximal such that $H^0(\mathcal{E} \otimes \mathcal{O}_C(-i)) \neq 0$. From the exact sequence $0 \rightarrow \mathcal{E}(-C-iD) \rightarrow \mathcal{E}(-iD) \rightarrow \mathcal{E} \otimes \mathcal{O}_C(-i) \rightarrow 0$ we get $H^0(\mathcal{E}(-iD)) = 0 \Rightarrow H^0(\mathcal{E}(-C-iD)) = 0$, twisting the sequence several times by $\mathcal{O}(-C)$ gives $H^0(\mathcal{E}(-nC-iD)) = 0$ for all $n \geq 0$ which is impossible since $\mathcal{O}(-C)$ is very ample relative to X . Hence we have a non-zero section $s \in H^0(\mathcal{E}(-iD))$ which defines a short exact sequence of locally free sheaves

$$0 \rightarrow \mathcal{O}(iD) \xrightarrow{s} \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$$

by the maximality of i . The lemma follows by induction on the rank since one from the maximality of i gets $\text{Ext}_X^1(\mathcal{F}, \mathcal{O}(iD)) = 0$. \square

Remark 1. If $\text{char } k = 0$, then X is the cyclic quotient singularity defined by the action of the cyclic group $G = \langle \zeta \cdot \text{id}_{k^2} \rangle$ on \mathbb{A}^2 , where ζ is a primitive m^{th} root of unity in k , see [19]. Moreover; there is a correspondence between the irreducible representations of G and the indecomposable reflexive \mathcal{O}_X -modules where $M_i = (k[u, v] \otimes_k \xi_i)^G$ if the irreducible representation ξ_i is given by $\zeta \cdot \text{id}_{k^2} \mapsto \zeta^i$ for $0 \leq i \leq m-1$, see [25].

Note that by adjunction $c_1(\omega_{\tilde{X}}) \cdot C = -C^2 - 2 = m - 2$ thus $\omega_{\tilde{X}} = \mathcal{O}_{\tilde{X}}((m-2)D)$ and since $\pi_*\omega_{\tilde{X}} = \omega_X$ for all rational surface singularities, we get $\omega_X = M_{m-2}$. We say that a locally free sheaf \mathcal{E} on \tilde{X} is *semi-full* (over a reflexive module M) if $R^1\pi_*\mathcal{E} = 0$ and $H^0(X, \pi_*\mathcal{E}) \cong M$. By [16, 1.8] there are natural embeddings $\tilde{M} \subseteq \mathcal{E} \subseteq \tilde{M}^\omega$ where $M^\omega = \text{Hom}_X(M, \omega_X)$. We have $\tilde{M}_i^\omega = \mathcal{O}_{\tilde{X}}((m-2-i)D)$ for $0 \leq i \leq m-1$ and $H^0(X, \pi_*(\mathcal{E}^\omega)) = M^\omega$ [10], hence we get $M_i^\omega = M_{m-2-i}$ for

$0 \leq i \leq m-2$ and $M_{m-1}^\omega = M_{m-1}$. We obtain

$$\widetilde{M}_i^\omega = \begin{cases} \widetilde{M}_i & \text{if } 0 \leq i \leq m-2 \\ \mathcal{O}_{\widetilde{X}}(-D) & \text{if } i = m-1. \end{cases}$$

Dividing the two inclusions $\widetilde{M} \subseteq \mathcal{E} \subseteq \widetilde{M}^\omega$ with \widetilde{M} give inclusions of sheaves $0 \subseteq \overline{\mathcal{E}} \subseteq \mathcal{O}_C(-1)^r$ on C , hence if \mathcal{E} is semi-full, with $\pi_*\mathcal{E} \cong M = \bigoplus_{i=0}^{m-1} M_i^{n_i}$, then (by the proof of Lemma 1) $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{G}$ where $\mathcal{F} = \bigoplus_{i=0}^{i=m-2} \mathcal{O}(iD)^{n_i}$ and $\mathcal{G} = \mathcal{O}(-D)^s \oplus \mathcal{O}((m-1)D)^{r-s}$ ($r = n_{m-1}$), this notation will be fixed.

Let \mathbf{Hens}_k be the category of local, Henselian k -algebras \mathcal{O}_S with residue field k . The deformation functor $\text{Def}_M : \mathbf{Hens}_k \rightarrow \mathbf{Sets}$ associates to \mathcal{O}_S the set of equivalence classes of *deformations* of M to \mathcal{O}_S . A deformation (or flat lifting) of M to \mathcal{O}_S is an $(\mathcal{O}_X \otimes_k \mathcal{O}_S)^{\text{h}}$ -module M_S , flat as \mathcal{O}_S -module together with an $(\mathcal{O}_X \otimes_k \mathcal{O}_S)^{\text{h}}$ -linear map $\pi : M_S \rightarrow M$ with $\pi \otimes_{\mathcal{O}_S} k : M_S \otimes_{\mathcal{O}_S} k \xrightarrow{\sim} M$. Two deformations are equivalent if they are isomorphic over M . Maps are induced by tensorisation. If the module is of finite type over an *algebraic* ring, i.e. the Henselisation of a k -algebra essentially of finite type, such that the locus where M is not free is of finite length, then, using [5] and [9, Thm. 3], it is shown in [24] and in [16] that there exists a *versal family* (R, M_R) for Def_M where in particular R is algebraic. We fix such a versal family where we assume that the Zariski tangent space is of minimal dimension at the central point and put $X_R = \text{Spec } \mathcal{O}_{X \times R}^{\text{h}}$. Moreover; since Def_M is a functor locally of finite presentation, there exists a *germ* representing (R, M_R) , i.e. an affine k -pointed k -scheme R^{ft} of finite type and an $\mathcal{O}_{R^{\text{ft}}}$ -flat family of reflexive modules $M_{R^{\text{ft}}}$, finitely generated as $\mathcal{O}_{X \times R^{\text{ft}}}$ -module, such that the Henselisation at the k -point gives (R, M_R) .

Definition 1. If M and N are two reflexive modules on a surface singularity, let $\text{Loc}(N)$ be the set of k -points $t \in R^{\text{ft}}(k)$ such that the pullback M_t of $M_{R^{\text{ft}}}$ to t is isomorphic to N . Then M *locally deforms to* N , denoted $M \dashrightarrow N$, if the Zariski closure $\overline{\text{Loc}}(N)$ strictly contains the central k -point t_0 corresponding to M . If, possibly after restricting to a Zariski open set in R^{ft} containing t_0 , the pullback of $M_{R^{\text{ft}}}$ to $\overline{\text{Loc}}(N) \setminus \{t_0\}$ is non-empty and only contains N as k -fibres, then $\overline{\text{Loc}}(N)$ is called an *absolute minimal stratum* of R^{ft} and the local deformation of M to N is called *minimal*.

It follows that the relation \dashrightarrow is independent of choice of germ, and by openness of versality [16, 2.13] it follows that the local deformation relation is *transitive*.

In [16] Ishii introduces a sub-functor $\text{Def}'_M \subseteq \text{Def}_M$ of deformations such that the induced deformation of the determinant bundle of M_S restricted to the regular locus U is trivial. Ishii shows that there is a versal family $(R', M_{R'})$ for Def'_M and that $R' \cong R_{\text{red}}$.

Let $d \in \text{Pic } \widetilde{X}$, then the *Ishii functor* of semi-full sheaves with isomorphism is defined as follows:

Definition 2 (A. Ishii [16, 4.2]). Let the functor

$$(1) \quad \mathcal{F}_{M_{R'}}^d : \text{Sch}_{R'} \rightarrow \mathbf{Sets}$$

for any $(\psi_S : S \rightarrow R')$ in $\text{Sch}_{R'}$ be given as the set $\mathcal{F}_{M_{R'}}^d(\psi_S : S \rightarrow R')$ of equivalence classes of pairs $(\mathcal{E}_S, \varphi_S)$ where

- i) \mathcal{E}_S is a locally free sheaf on $\widetilde{X}_S = \widetilde{X} \times_X X_S$,
- ii) $R^1\pi_*\mathcal{E}_t = 0$ and $c_1(\mathcal{E}_t) = d$ for all pullbacks \mathcal{E}_t of \mathcal{E}_S to k -points $t \in S(k)$,
- iii) $\varphi_S : \pi_{S*}\mathcal{E}_S \xrightarrow{\sim} \psi_S^*M_{R'}$ on $X_S = X_{R'} \times_{R'} S$.

Two pairs $(\mathcal{E}_S, \varphi_S)$ and $(\mathcal{E}'_S, \varphi'_S)$ are equivalent if there is an isomorphism $\tau : \mathcal{E}_S \xrightarrow{\sim} \mathcal{E}'_S$ such that $\pi_{S*}(\tau) = (\varphi'_S)^{-1}\varphi_S$.

Ishii's main theorem [16, 4.9] states that the functor $\mathcal{F}_{M_{R'}}^d$ is represented by a R' -scheme $\psi_{F^d} : F^d = F_{M_{R'}}^d \rightarrow R'$ which is projective over R' , regular, non-empty for a finite set of Chern classes d , and their images $\{R^d\}$ in R' constitutes a filtration of R' . There is an isomorphism between the locus S^d in R' of reflexive modules with first Chern class equal to d and the open set in $F_{M_{R'}}^d$ corresponding to full sheaves.

We may take the structure map ψ_S into the functor and define $\mathcal{T}_{M_{R'}}^d(S)$ for a k -scheme S as equivalence classes of tuples $(\mathcal{E}_S, \psi_S : S \rightarrow R', \varphi_S)$ where $(\mathcal{E}_S, \varphi_S)$ defines an element in $\mathcal{F}_{M_{R'}}^d(\psi_S : S \rightarrow R')$. One can check that $\mathcal{T}_{M_{R'}}^d$ is representable if and only if $\mathcal{F}_{M_{R'}}^d$ is representable. We extend $\mathcal{T}_{M_{R'}}^d$ by allowing the range of ψ_S to be the non-reduced R and define $\mathcal{T}^d = \mathcal{T}_{M_R}^d$ as equivalence classes of triples $(\mathcal{E}_S, \psi_S : S \rightarrow R, \varphi_S)$ with $(\mathcal{E}_S, \varphi_S)$ as in Definition 2 with R substituting R' . *A priori* $\mathcal{T}_{M_{R'}}^d \subseteq \mathcal{T}_{M_R}^d$, but for X a rational cone we show in Theorem 1 that $\mathcal{T}_{M_R}^d$ is represented by a regular scheme T^d , hence $\psi_{T^d} : T^d \rightarrow R$ factors through R' and $F^d = T^d$ as k -schemes.

By “module” we will usually mean “reflexive module”. As a convention the first projection will usually be denoted p , like in $p : \tilde{X} \times S \rightarrow \tilde{X}$, and the second projection q .

3. REPRESENTING THE ISHII FUNCTOR

Theorem 1 states that the representing space for $\mathcal{T}_{M_R}^d$ is given as the total space of a vector bundle of relative extensions of a locally free sheaf \mathcal{E}_A with itself over a Grassmannian A . Let $\mathcal{A} = \mathcal{A}_M^d$ be the sub-functor of \mathcal{T}^d of tuples $(\mathcal{E}_S, \psi_S : S \rightarrow R, \varphi_S)$ where ψ_S factorises through $\text{Spec } k$, i.e. is trivial. In Proposition 1 we show that \mathcal{A}_M^d is represented by $A = A_M^d$ with a universal locally free sheaf \mathcal{E}_A on $\tilde{X} \times A$ and in Proposition 2 we give a natural embedding of the sheaf of relative extensions of \mathcal{E}_A with itself into the trivial pullback of $\text{Ext}_X^1(M, M)$ to A .

Proposition 1. *Let $d = c_1(\tilde{M}) + sC$ with $0 \leq s \leq r$ where r is the multiplicity of M_{m-1} in M . Then the functor \mathcal{A}_M^d is represented by the Grassmannian $A = \text{Grass}(s, r)$.*

Proof. There is a natural isomorphism $\alpha : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$ valid for all rational surface singularities where $\mathcal{B}(S)$ is defined as the set of equivalence classes of embeddings $\iota_S : \mathcal{E}_S \hookrightarrow p^* \tilde{M}^{\omega}$ where \mathcal{E}_S is a locally free coherent sheaf with $c_1(\mathcal{E}_t) = d$, $R^1 \pi_* \mathcal{E}_t = 0$ and $\pi_* \mathcal{E}_t \xrightarrow{\sim} M$ for all $t \in S(k)$ and with $\iota_S \sim \iota'_S$ if $\text{im } \iota_S = \text{im } \iota'_S$. It is not obvious that \mathcal{B} is a functor, i.e. whether a pullback of ι_S will be an injective map, this is however a consequence of the following argument. Given $(\mathcal{E}_S, \varphi_S)$ in $\mathcal{A}(S)$ the inclusion $\iota_S = \alpha(\mathcal{E}_S, \varphi_S)$ is the following composition

$$(2) \quad \mathcal{E}_S \xrightarrow{\sim} (\mathcal{E}_S^\omega)^\omega \hookrightarrow (\pi_S^* \pi_{S*} \mathcal{E}_S^\omega)^\omega \\ \cong (\pi_{S*}(\mathcal{E}_S^\omega))^\omega \xrightarrow[\cong]{(c^{-1})^\omega} ((\pi_{S*}(\mathcal{E}_S)^\omega)^\omega)^\omega \xrightarrow[\cong]{\tilde{\varphi}_S^\omega} p^* \tilde{M}^{\omega} \cong p^* \tilde{M}^{\omega}$$

where all maps are canonical except $\tilde{\varphi}_S^\omega$. The first map is an isomorphism since \mathcal{E}_S is locally free. For the second one the cokernel of the natural map $\pi_S^* \pi_{S*} \mathcal{E}_S^\omega \rightarrow \mathcal{E}_S^\omega$ has support on $C \times S$, dualising in $p^* \omega_{\tilde{X}}$ hence gives an inclusion. The next isomorphism follows similarly. The canonical isomorphism $\pi_* \omega_{\tilde{X}} \rightarrow \omega_X$ induces the isomorphism $c : \pi_{S*}(\mathcal{E}_S^\omega) \rightarrow (\pi_{S*} \mathcal{E}_S)^\omega$. The two last isomorphisms are clear. The construction of ι_S from φ_S is clearly functorial, hence α is well defined: An isomorphism $\theta : \mathcal{E}_S \rightarrow \mathcal{E}'_S$ compatible with φ_S and φ'_S gives $\tilde{\varphi}_S^\omega = \tilde{\varphi}'_S^\omega \circ ((\pi_{S*}(\theta)^\omega)^\omega)^\omega$ and $\text{im } \iota_S = \text{im } \iota'_S$.

The inverse $\beta : \mathcal{B} \rightarrow \mathcal{A}$ to α is given by

$$(\iota : \mathcal{E}_S \hookrightarrow p^* \widetilde{M}^{\omega}) \mapsto (\mathcal{E}_S, \pi_{S*}(\iota) : \pi_{S*}(\mathcal{E}_S) \rightarrow \pi_{S*} p^* \widetilde{M}^{\omega} = p^* M),$$

$\iota_* = \pi_{S*}(\iota)$ is an isomorphism by definition of \mathcal{B} . In particular; \mathcal{B} is a functor.

Now the general idea is to map $\iota_S : \mathcal{E}_S \hookrightarrow p^* \widetilde{M}^{\omega}$ in $\mathcal{B}(S)$ to the induced embedding $\bar{\iota}_S : \bar{\mathcal{E}}_S = \mathcal{E}_S / \iota_S^{(-1)}(p^* \widetilde{M}) \hookrightarrow p^*(\widetilde{M}^{\omega} / \widetilde{M})$ with support on $C \times S$ so that one is left to study a Quot-functor on the exceptional fibre. Here $\iota_S^{(-1)}$ is defined as follows. Pulling the inverse $(\iota_*)^{-1}$ back to $\pi_S^*(\iota_*^{-1}) : p^* \pi^* M \xrightarrow{\sim} \pi_S^* \pi_{S*} \mathcal{E}_S$ followed by the canonical $\pi_S^* \pi_{S*} \mathcal{E}_S \rightarrow \mathcal{E}_S$ induces a map $\iota_S^{(-1)} : p^* \widetilde{M} \rightarrow \mathcal{E}_S$, functorial in φ_S , the composition $\iota_S \circ \iota_S^{(-1)} : p^* \widetilde{M} \rightarrow p^* \widetilde{M}^{\omega}$ is the natural inclusion, [15, 1.8], and hence $\iota_S^{(-1)}$ is an inclusion. In our case there is a canonical splitting $\mathcal{E}_S = p^* \mathcal{F} \oplus \mathcal{G}_S$ and $\bar{\mathcal{E}}_S = \mathcal{G}_S / \iota_S^{(-1)}(p^* \mathcal{O}_{\tilde{X}}((m-1)D)^r)$. Since $\iota_S^{(-1)}$ is functorial in S , the short exact sequence $0 \rightarrow p^* \widetilde{M} \xrightarrow{\iota_S^{(-1)}} \mathcal{E}_S \rightarrow \bar{\mathcal{E}}_S \rightarrow 0$ is natural for pullbacks of S , hence $\bar{\mathcal{E}}_S$ is S -flat. It follows that the inclusion $\bar{\iota}_S : \bar{\mathcal{E}}_S \hookrightarrow p^* \mathcal{O}_C(-1)^r$ is natural for pullbacks of S , hence $\bar{\mathcal{E}}_t$ is a locally free sheaf for $t \in S(k)$. From the inclusion $\bar{\iota}_S$ we have $H^0(C, \bar{\mathcal{E}}_t) = 0$, and since \mathcal{E}_t is semi-full $0 = H^1(\tilde{X}, \mathcal{E}_t) \rightarrow H^1(C, \bar{\mathcal{E}}_t)$, hence $\bar{\mathcal{E}}_t \cong \mathcal{O}_C(-1)^{s'}$. We claim that $s' = s$. If $\rho = \text{rk } \widetilde{M}$ we choose $\rho - 1$ generic sections in $H^0(\tilde{X}, \widetilde{M}) \xrightarrow{\sim} H^0(\tilde{X}, \mathcal{E}_t)$ which define inclusions of the trivial sheaf $\mathcal{O}_{\tilde{X}}^{\rho-1}$ in \widetilde{M} and in \mathcal{E}_t to obtain representatives of the first Chern class as the cokernel, see [2]. There is a commutative diagram of sheaves on \tilde{X} with two horizontal and two vertical short exact sequences from which the claim follows:

$$(3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\tilde{X}}^{\rho-1} & \longrightarrow & \widetilde{M} & \longrightarrow & \mathcal{O}_{\tilde{X}}(c_1(\widetilde{M})) \longrightarrow 0 \\ & & \parallel & & \downarrow \iota_t^{(-1)} & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_{\tilde{X}}^{\rho-1} & \longrightarrow & \mathcal{E}_t & \longrightarrow & \mathcal{O}_{\tilde{X}}(c_1(\widetilde{M}) + sC) \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \bar{\mathcal{E}}_t & \xrightarrow{\cong} & \mathcal{O}_{sC}(c_1(\widetilde{M}) + sC) \end{array}$$

By Cohomology and Base Change [13, III.12.11], twisting by 1 and pushing down to S gives an embedding of a locally free sheaf of rank s ; $q_* \bar{\iota}_S(1) : q_* \bar{\mathcal{E}}_S(1) \hookrightarrow \mathcal{O}_S^r$, hence an element in $\text{Quot}_{k^r}^s(S)$.

For the inverse, let $0 \rightarrow \mathcal{V} \rightarrow \mathcal{O}_S^r \xrightarrow{\tau} \mathcal{O}_S^r / \mathcal{V} \rightarrow 0$ be an element in $\text{Quot}_{k^r}^s(S)$. Let $\mathcal{G}_S = \ker \eta \otimes \tau$;

$$0 \rightarrow \mathcal{G}_S \xrightarrow{g_S} p^* \mathcal{O}_{\tilde{X}}(-D) \otimes q^* \mathcal{O}_S^r \xrightarrow{\eta \otimes \tau} p^* \mathcal{O}_C(-1) \otimes q^* \mathcal{O}_S^r / \mathcal{V} \rightarrow 0$$

where η is the quotient map in the short exact sequence

$$(4) \quad 0 \rightarrow \mathcal{O}_{\tilde{X}}(-C - D) \rightarrow \mathcal{O}_{\tilde{X}}(-D) \xrightarrow{\eta} \mathcal{O}_C(-1) \rightarrow 0.$$

Since $\text{Tor}_i^{\tilde{X} \times S}(p^* \mathcal{O}_C(-1) \otimes q^* \mathcal{O}_S^r / \mathcal{V}, -) = 0$ for $i \geq 2$, \mathcal{G}_S is a coherent locally free $\mathcal{O}_{\tilde{X} \times S}$ -sheaf. Together with the trivial embedding of $p^* \mathcal{F}$, g_S gives the element $\iota_S : p^* \mathcal{F} \oplus \mathcal{G}_S \hookrightarrow p^* \widetilde{M}^{\omega}$ in $\mathcal{B}(S)$. One checks that this gives an inverse. \square

Let

$$(5) \quad \iota = (\text{id} \oplus g) : \mathcal{E}_A = p^* \mathcal{F} \oplus \mathcal{G}_A \hookrightarrow p^* \mathcal{F} \oplus p^* \mathcal{O}_{\tilde{X}}(-D)^r = p^* \widetilde{M}^{\omega}$$

be the universal embedded locally free sheaf on $\tilde{X} \times A$ where A is the Grassmannian representing \mathcal{A} . Let $\tau : \mathcal{O}_A^r \rightarrow \mathcal{Q}$ be the universal quotient map on the Grassmannian.

Lemma 2. *There is a natural short exact sequence of locally free sheaves on $\tilde{X} \times A$*

$$(6) \quad 0 \rightarrow p^* \mathcal{O}_{\tilde{X}}(-D) \otimes q^* \mathcal{S} \rightarrow \mathcal{G}_A \rightarrow p^* \mathcal{O}_{\tilde{X}}(-C-D) \otimes q^* \mathcal{Q} \rightarrow 0$$

which is locally split on A .

Proof. The short exact sequence

$$(7) \quad 0 \rightarrow \mathcal{G}_A \xrightarrow{g} p^* \mathcal{O}_{\tilde{X}}(-D) \otimes q^* \mathcal{O}_A^r \xrightarrow{\eta \otimes \tau} p^* \mathcal{O}_C(-1) \otimes q^* \mathcal{Q} \rightarrow 0$$

defines \mathcal{G}_A . Pull the canonical short exact sequence

$$(8) \quad 0 \rightarrow p^* \mathcal{O}_{\tilde{X}}(-D) \otimes q^* \mathcal{S} \rightarrow p^* \mathcal{O}_{\tilde{X}}(-D) \otimes q^* \mathcal{O}_A^r \rightarrow p^* \mathcal{O}_{\tilde{X}}(-D) \otimes q^* \mathcal{Q} \rightarrow 0$$

back along the inclusion $p^* \mathcal{O}_{\tilde{X}}(-C-D) \otimes q^* \mathcal{Q} \rightarrow p^* \mathcal{O}_{\tilde{X}}(-D) \otimes q^* \mathcal{Q}$. Then g is the induced inclusion $\mathcal{G}_A \rightarrow p^* \mathcal{O}_{\tilde{X}}(-D) \otimes q^* \mathcal{O}_A^r$, and the short exact sequence (6) is obtained. A local splitting of the universal inclusion $\mathcal{S} \hookrightarrow \mathcal{O}_A^r$ gives a local splitting of (8) which induces the local splitting of $p^* \mathcal{O}_{\tilde{X}}(-D) \otimes q^* \mathcal{S} \hookrightarrow \mathcal{G}_A$. \square

Assume \mathcal{M} is a quasi-coherent sheaf on $\tilde{X} \times S$. Let $\mathcal{E}xt_{\tilde{X} \times S/S}^*(\mathcal{M}, -)$ denote the derived functor of $q_* \mathcal{H}om_{\tilde{X} \times S}(\mathcal{M}, -) : \text{Mod}_{\tilde{X} \times S} \rightarrow \text{Mod}_S$, [16, 5.4]. There is a first quadrant cohomological spectral sequence

$$(9) \quad E_2^{ij} = R^i q_* \mathcal{E}xt_{\tilde{X} \times S}^j(\mathcal{M}, -) \Rightarrow \mathcal{E}xt_{\tilde{X} \times S/S}^*(\mathcal{M}, -).$$

If \mathcal{M} is locally free the spectral sequence degenerates to

$$(10) \quad R^i q_* \mathcal{H}om_{\tilde{X} \times S}(\mathcal{M}, -) = \mathcal{E}xt_{\tilde{X} \times S/S}^i(\mathcal{M}, -).$$

Proposition 2. *There is a natural injective homomorphism of \mathcal{O}_A -sheaves*

$$\mathcal{E}xt_{\tilde{X} \times A/A}^1(\mathcal{E}_A, \mathcal{E}_A) \hookrightarrow \text{Ext}_X^1(M, M) \otimes_k \mathcal{O}_A.$$

Proof. The universal inclusion $\iota : \mathcal{E}_A \hookrightarrow p^* \widetilde{M}^{\omega}$ and the induced inclusion $\iota^{(-1)} : p^* \widetilde{M} \hookrightarrow \mathcal{E}_A$ (cf. the proof of Proposition 1) give two maps

$$(11) \quad \mathcal{E}xt_{\tilde{X} \times A/A}^1(\mathcal{E}_A, \mathcal{E}_A) \xrightarrow{(\iota^{(-1)})^*} \mathcal{E}xt_{\tilde{X} \times A/A}^1(p^* \widetilde{M}, \mathcal{E}_A) \xrightarrow{\iota_*} \mathcal{E}xt_{\tilde{X} \times A/A}^1(p^* \widetilde{M}, p^* \widetilde{M}^{\omega}).$$

In Theorem 2 we prove in a more precise statement that this composition is an inclusion. By (10)

$$\begin{aligned} \mathcal{E}xt_{\tilde{X} \times A/A}^1(p^* \widetilde{M}, p^* \widetilde{M}^{\omega}) &\cong R^1 q_* \mathcal{H}om_{\tilde{X} \times A}(p^* \widetilde{M}, p^* \widetilde{M}^{\omega}) \\ &\cong R^1 q_* p^* \mathcal{H}om_{\tilde{X}}(\widetilde{M}, \widetilde{M}^{\omega}) \cong \text{Ext}_X^1(\widetilde{M}, \widetilde{M}^{\omega}) \otimes_k \mathcal{O}_A. \end{aligned}$$

The proposition then follows from composing with the following isomorphism, valid for all rational surface singularities

$$(12) \quad \pi_* : \text{Ext}_X^1(\widetilde{M}, \widetilde{M}^{\omega}) \xrightarrow{\sim} \text{Ext}_X^1(M, M).$$

The latter is proved as in [15, 3.5] with $^{\vee}$ changed to $^{\omega}$. \square

Set $E = \text{Spec Sym}_k(\text{Ext}_X^1(M, M)^*)$ and let E^h be the Henselisation of E at the origin. There is an embedding of R into E^h which is not canonical; two embeddings differ by an automorphism of E^h which induces the identity on the Zariski tangent space. To accommodate this inconvenience we consider the functor $\mathcal{R}^d = \mathcal{R}_{M_R}^d$ which is defined to be the image of \mathcal{T}^d in $\text{Hom}_{\text{Sch}_k}(-, R)$ under the map $(\mathcal{E}_S, \psi_S, \varphi_S) \mapsto \psi_S$.

Theorem 1. *Let (R, M_R) be the versal family for the reflexive module M on the cone X over the rational normal curve of degree m , and let $d = c_1(\widetilde{M}) + sC \in \text{Pic } \widetilde{X}$ with $0 \leq s \leq r$ where r is the multiplicity of M_{m-1} in M . If \mathcal{E}_A is the universal sheaf on $\widetilde{X} \times A$ from (5), then the functor $\mathcal{T}_{M_R}^d$ is represented by the k -scheme*

$$T^d = \mathbf{Spec} \text{Sym}_{\mathcal{O}_A}(\mathcal{E}xt_{\widetilde{X} \times A/A}^1(\mathcal{E}_A, \mathcal{E}_A)^\vee) \times_E E^h$$

where the map to E is induced by the inclusion in Proposition 2 and where the Grassmannian $A = \text{Grass}(s, r)$ represents the functor \mathcal{A}_M^d .

Moreover; let R^d be the image of T^d in E^h and let $\Psi^d : T^d \rightarrow R^d$ be the induced map. Then R^d represents $\mathcal{R}_{M_R}^d$ and $\mathcal{T}_{M_R}^d \rightarrow \mathcal{R}_{M_R}^d$ is induced by Ψ^d .

Proof. Let T and R^d in the following argument be the T^d and the R^d “without the h ” unless pullback to E^h is called for. The proof has two parts.

- (1) Construction of an element $(\mathcal{E}_T, \psi_T, \varphi_T)$ in $\mathcal{T}^d(T)$.
- (2) Show that this element is universal for \mathcal{T}^d .

1. There is a covering of A by open affines $\{V_i\}$ such that $\mathcal{E}xt_{\widetilde{X} \times A/A}^1(\mathcal{E}_A, \mathcal{E}_A)|_{V_i}$ is a free $\mathcal{O}_{V_i} = \mathcal{O}_A(V_i)$ -sheaf for all i by Lemma 2 and (10). Let $\mathcal{E}_{V_i} := \mathcal{E}_A|_{V_i}$ then

$$\Gamma(V_i, \mathcal{E}xt_{\widetilde{X} \times A/A}^1(\mathcal{E}_A, \mathcal{E}_A)) \cong H^1(\widetilde{X} \times V_i, \mathcal{E}nd_{\widetilde{X} \times V_i}(\mathcal{E}_{V_i})) =: H^1$$

with a $B_i := \mathcal{O}_A(V_i)$ -basis $\{\eta_1^{(i)}, \dots, \eta_n^{(i)}\}$. There is a covering $\{U_0, U_1\}$ of \widetilde{X} by affine open sub-schemes. Let $U_{01} = U_0 \cap U_1$, then \mathcal{E}_{V_i} is defined by a transition map $\theta \in \Gamma(U_{01} \times V_i, \mathcal{E}nd_{\widetilde{X} \times A}(\mathcal{E}_A))$ (e.g. by Lemma 2 since $\text{Pic } \widetilde{X}$ is generated by (any) D). Define \mathcal{E}_i on

$$T_i = \text{Spec} \text{Sym}_{B_i}((H^1)^\vee) = B_i[t_1, \dots, t_n]$$

by extending the transition map to $\tilde{\theta} = \theta + \sum_{j=1}^n \eta_j^{(i)} t_j$ where $t_j = t_j^{(i)}$ in $(H^1)^\vee$ is B_i -dual to $\eta_j^{(i)}$. Remark that $\tilde{\theta} \bmod (t_1, \dots, t_n)^2$ gives the universal lifting of \mathcal{E}_{V_i} to $B_i[t_1, \dots, t_n]/(t_1, \dots, t_n)^2$. In fact the \mathcal{E}_i glue together on the full scheme giving \mathcal{E}_T , since $\tilde{\theta}$ is independent of choice of basis, which follows since the t_j have degree one in the B_i -linear grading of T_i , and this grading is preserved by localisation of B_i . By Lemma 2, after pullback to E^h , $R^1\pi_*(\mathcal{E}_T \otimes_{\mathcal{O}_T} k(t)) = 0$ for all $t \in T(k)$.

In order to find $\psi_T : T \rightarrow R$ and $\varphi_T : \pi_{T*}\mathcal{E}_T \xrightarrow{\sim} \psi_T^*M_R$ we first show that $\tilde{\alpha}_*\mathcal{E}_T$ is a locally free sheaf, hence H -flat, where $\tilde{\alpha} : \widetilde{X} \times T \rightarrow \widetilde{X} \times H$ is the pullback of $\rho : T \rightarrow H = \text{Spec} \text{Sym}_k H^0(T, \mathcal{O}_T)$. On U_i , C is defined by v_i , pushout of (6) restricted to U_i along the isomorphism $p^*\mathcal{O}_{U_i}(-D) \otimes q^*\mathcal{S} \xrightarrow{\cdot v_i \otimes 1} p^*\mathcal{O}_{U_i}(-C-D) \otimes q^*\mathcal{S}$ gives the exact sequence $0 \rightarrow p^*\mathcal{O}_{U_i}(-C-D) \otimes q^*\mathcal{S} \rightarrow \mathcal{G}'_A \rightarrow p^*\mathcal{O}_{U_i}(-C-D) \otimes q^*\mathcal{Q} \rightarrow 0$ with $\mathcal{G}'_A \cong \mathcal{G}_A$. The natural maps

$$\begin{aligned} \text{Ext}_{U_i \times A}^1(\mathcal{O}_{U_i}(-D) \boxtimes \mathcal{Q}, \mathcal{O}_{U_i}(-D) \boxtimes \mathcal{S}) &\cong \mathcal{O}_{U_i} \otimes_k \text{Ext}_A^1(\mathcal{Q}, \mathcal{S}) \rightarrow \\ \text{Ext}_{U_i \times A}^1(\mathcal{O}_{U_i}(-C-D) \boxtimes \mathcal{Q}, \mathcal{O}_{U_i}(-D) \boxtimes \mathcal{S}) &\cong \mathcal{O}_{U_i}(C) \otimes_k \text{Ext}_A^1(\mathcal{Q}, \mathcal{S}) \xrightarrow{\cdot v_i \otimes 1} \\ \text{Ext}_{U_i \times A}^1(\mathcal{O}_{U_i}(-C-D) \boxtimes \mathcal{Q}, \mathcal{O}_{U_i}(-C-D) \boxtimes \mathcal{S}) &\cong \mathcal{O}_{U_i} \otimes_k \text{Ext}_A^1(\mathcal{Q}, \mathcal{S}) \end{aligned}$$

takes the canonical sequence on A tensor $\mathcal{O}_{U_i}(-D)$ to (6) and further on to the canonical sequence on A tensor $\mathcal{O}_{U_i}(-C-D)$, therefore $\mathcal{G}_{A|U_i \times A} \cong \mathcal{O}_{U_i}(-D) \boxtimes \mathcal{O}_A^r$ and hence $\mathcal{E}_{A|U_i \times A}$ is free. Thus the tensor product pre-sheaf $\mathcal{E}_{A|U_i \times A} \otimes_{\mathcal{O}_A} \mathcal{O}_T$ is a sheaf and $\mathcal{E}_T|_{U_i \times T} = \mathcal{E}_{A|U_i \times A} \otimes_{\mathcal{O}_A} \mathcal{O}_T$. Therefore

$$\tilde{\alpha}_*\mathcal{E}_T|_{U_i \times H}(U_i \times H) \cong \mathcal{E}_{A|U_i \times A}(U_i \times A) \otimes_{\mathcal{O}_A(A)} \mathcal{O}_T(T) \cong \mathcal{O}_{U_i}(U_i)^{\text{rk}(\mathcal{E})} \otimes_k \mathcal{O}_H.$$

Remark that H is of finite type since $T \rightarrow E$ is a projective morphism which follows from Propositions 1, 2. Since $\tilde{\alpha}_*\mathcal{E}_T$ is locally free, we have that $\pi_{H*}\tilde{\alpha}_*\mathcal{E}_T$ is an \mathcal{O}_H -flat family of reflexive modules on X by [15, 3.4], with central fibre M .

After Henselisation there is, by versality of (R, M_R) , a map $\gamma : H \rightarrow R$ and an isomorphism $\bar{f} : (\pi_H \tilde{\alpha})_* \mathcal{E}_T \xrightarrow{\sim} (\text{id} \times \gamma)^* M_R$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{X} \times T & \xrightarrow{\tilde{\alpha}} & \tilde{X} \times H \\ \downarrow \pi_T & & \downarrow \pi_H \\ X \times T & \xrightarrow{\alpha} & X \times H \end{array}$$

Let $f : \alpha^* \alpha_*(\pi_T)_* \mathcal{E}_T \xrightarrow{\sim} ((\text{id} \times \gamma)\alpha)^* M_R$ be the pullback of \bar{f} by α to $X \times T$. We claim that the canonical map $\theta : \alpha^* \alpha_*(\pi_T)_* \mathcal{E}_T \rightarrow \pi_{T*} \mathcal{E}_T$ is an isomorphism and we put $\varphi_T = f\theta^{-1}$. For the surjectivity of θ , there is a natural map $u : (\pi_{T*} \mathcal{E}_T) \otimes_{\mathcal{O}_T} \mathcal{O}_A \rightarrow \pi_{A*}(\mathcal{E}_T \otimes_{\mathcal{O}_T} \mathcal{O}_A) \cong M \otimes_k \mathcal{O}_A$. To show that u is an isomorphism we consider the sheafified Čech complex $0 \rightarrow \mathcal{E}_T \rightarrow \mathcal{C}^0 \rightarrow \mathcal{C}^1 \rightarrow 0$ on $\tilde{X} \times T$, applying π_{T*} gives a short exact sequence since $R^1 \pi_{T*} \mathcal{E}_T = 0$ by Cohomology and Base Change [13, III.12.11]. Applying $-\otimes_{\mathcal{O}_T} \mathcal{O}_A$ leaves the sequence exact since $\pi_{T*} \mathcal{C}^1$ is flat as \mathcal{O}_T -module. Since we also have $(\alpha^* \alpha_*(\pi_T)_* \mathcal{E}_T) \otimes_{\mathcal{O}_T} \mathcal{O}_A \cong M \otimes_k \mathcal{O}_A$, $\theta \otimes_{\mathcal{O}_T} \mathcal{O}_A$ is an isomorphism and $\text{coker } \theta = 0$. Injectivity follows from $(\ker \theta) \otimes_{\mathcal{O}_T} \mathcal{O}_A \cong \mathcal{T}or_1^{\mathcal{O}_T}(\mathcal{O}_A, \pi_{T*} \mathcal{E}_T) = 0$ ([15, 3.4]). With $\psi_T = \gamma \circ \rho$, we get a T -point $\xi = [(\mathcal{E}_T, \psi_T, \varphi_T)] \in \mathcal{T}^d(T)$.

2. Restricting to the reduced locus $R' \subseteq R$ (see comments at the end of Section 2), we assume $\mathcal{T}_{M_R'}^d$ is represented by the regular scheme F^d which is projective over R' [16, 4.9]. Since T is regular, ψ_T factors through R' and $\xi \in \mathcal{T}^d(T)$ induce a map $f : T \rightarrow F^d$ of R' -schemes. The central fibre of T and of F^d is A , and (after the pullback to E^h) all k -rational points of T are contained in A . Let $t = [(\mathcal{E}, \varphi)] \in \mathcal{A}^d(k) = \mathcal{T}^d(k) = A(k)$ and let \mathcal{T}_t and \mathcal{A}_t be the associated deformation functors of \mathcal{T}^d and \mathcal{A}^d at t . There is a short exact sequence of k -vector spaces

$$(13) \quad 0 \rightarrow \mathcal{A}_t(k[\varepsilon]) \rightarrow \mathcal{T}_t(k[\varepsilon]) \rightarrow \text{Def}_{\mathcal{E}}(k[\varepsilon]) \rightarrow 0$$

which follows from [16, 4.11] since the composition is the trivial map: If $\iota : \mathcal{E} \hookrightarrow \widetilde{M}^{\omega}$ is the embedding (2) corresponding to φ , $\mathcal{A}_t(k[\varepsilon]) \rightarrow \text{Def}_{\mathcal{E}}(k[\varepsilon])$ is the composition $\text{Hom}_{\tilde{X}}(\mathcal{E}/\iota^{(-1)}(\widetilde{M}), \widetilde{M}^{\omega}/\iota(\mathcal{E})) \xrightarrow{\sim} \text{Hom}_{\tilde{X}}(\mathcal{E}, \widetilde{M}^{\omega}/\iota(\mathcal{E})) \xrightarrow{\delta} \text{Ext}_{\tilde{X}}^1(\mathcal{E}, \mathcal{E})$ of natural maps. The composition map factors through $\text{Ext}_{\tilde{X}}^1(\mathcal{E}/\iota^{(-1)}(\widetilde{M}), \mathcal{E})$ and since $\text{Ext}_{\tilde{X}}^1(\mathcal{E}, \mathcal{E}) \rightarrow \text{Ext}_{\tilde{X}}^1(\widetilde{M}, \mathcal{E})$ is injective by Proposition 2 ($R^1 q_*$ commutes with tensor products), it is trivial. From (13) and the construction of T it follows that f induces an isomorphism on the Zariski tangent spaces and hence $\hat{\mathcal{O}}_{F, f(t)} \xrightarrow{\sim} \hat{\mathcal{O}}_{T, t}$ for all $t \in T(k)$, by [12, 17.6.3] $T \rightarrow F$ is étale and since it is bijective on k -rational points it is an isomorphism.

One may also prove representability for the a priori larger $\mathcal{T}_{M_R}^d$ directly. Pro-representability of \mathcal{T}_t follows essentially as [20, 3.2], $\dim_k \mathcal{T}_t(k[\varepsilon]) < \infty$ is given by (13). The substantial part is to show that an isomorphism $\tau : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ of liftings of \mathcal{E} to S in \mathbf{Art}_k compatible with $\varphi_i : \pi_{S*} \mathcal{E}_i \xrightarrow{\sim} \psi_R^* M_R$ is uniquely determined, which follows by induction on the length of S : The set of liftings of τ in a small extension $S' \twoheadrightarrow S$ is a torsor over $\text{End}_{\tilde{X}}(\mathcal{E}) \otimes \ker(S' \twoheadrightarrow S)$, but $\text{End}_{\tilde{X}}(\mathcal{E}) = \text{End}_X(M)$ and $\pi_{S*} \tau$ is uniquely determined as $\varphi_2^{-1} \varphi_1$. Let $[(\mathcal{E}_{T'}, \psi_{T'}, \varphi_{T'})] \in \mathcal{T}^d(T')$ and denote by $A' \subseteq T'$ the closed fibre. By Proposition 1 there is a unique map $A' \rightarrow A$ which hence defines $T'(k) \rightarrow T(k)$ as a set map. By working locally on T' (in the étale topology) and applying Artin's Approximation Theorem [4, 2.2] one obtains a map $f : T' \rightarrow T$ of schemes over R , using the uniqueness-of-isomorphism argument above one shows that there is a unique isomorphism $f^* \mathcal{E}_T \xrightarrow{\sim} \mathcal{E}_{T'}$.

For the last part of Theorem 1 we show that $(\alpha \pi_T)_* \mathcal{E}_T$ is the pullback of an \mathcal{O}_{R^d} -flat $\mathcal{O}_{X \times R^d}$ -module M_{R^d} , hence $\gamma : H \rightarrow R$ may be chosen as a factorisation through R^d . Since $\tilde{\alpha}_* \mathcal{E}_{T|U_i \times H}$ is free and the transition matrix in fact is defined

over $\mathcal{O}_{U_i \times R^d}$ by construction of \mathcal{E}_T , $\tilde{\alpha}_* \mathcal{E}_T$ descends to a locally free $\tilde{X} \times R^d$ -module which is pushed down to an R^d -flat $X \times R^d$ -module M_{R^d} by [15, 3.4]. Remark that the various $t_j^{(i)}$ (in the construction of \mathcal{E}_T) are global sections in the image $H^0(\mathcal{O}_{E \times A}) \rightarrow H^0(\mathcal{O}_T)$, hence are contained in \mathcal{O}_{R^d} . We conclude that any $\psi_{T'} \in \mathcal{R}_{M_R}^d(T')$ factorises through R^d . \square

Remark 2. By versality of (R, M_R) and the last part of Theorem 1 one obtains R^d as a uniquely defined closed sub-scheme of R .

4. EQUATIONS FOR THE STRATA

Recall the map Ψ^d which is the embedding $T^d \hookrightarrow E^h \times A$ followed by the projection onto its image R^d in E^h . In Theorem 2 we calculate both sides of the inclusion in Proposition 2 and the inclusion itself and hence, by Theorem 1, obtain an explicit description of T^d and of the embedding in terms of canonical bundles and maps between them on the Grassmannian $A = \text{Grass}(s, r)$. Explicit equations are given in Corollary 1, and in Corollary 2 we obtain equations for the minimal strata $R^d \subseteq E^h$ ($s = 1$). An ideal-sheaf I_d on E^h which define $\Psi^d : T^d \rightarrow R^d$ as a blowing up is given in Corollary 3. With notation and assumptions as in Theorem 1 we have:

Theorem 2. *Let $\mathcal{E}_A = p^* \mathcal{F} \oplus \mathcal{G}_A$ be the universal sheaf on $\tilde{X} \times A$ in (5). The map $\text{Ext}_{\tilde{X} \times A/A}^1(\mathcal{E}_A, \mathcal{E}_A) \rightarrow \text{Ext}_X^1(M, M) \otimes_k \mathcal{O}_A$ stated in Proposition 2 gives an embedding $T^d \hookrightarrow E^h \times A$ and is the direct sum of the following four embeddings of coherent sheaves on A :*

- 1) $\text{Ext}_{\tilde{X} \times A/A}^1(p^* \mathcal{F}, p^* \mathcal{F}) = H^1(\text{End } \mathcal{F}) \otimes_k \mathcal{O}_A \xrightarrow{\cong} H^1(\mathcal{E}nd \mathcal{F}) \otimes_k \mathcal{O}_A$
- 2) $\text{Ext}_{\tilde{X} \times A/A}^1(p^* \mathcal{F}, \mathcal{G}_A) = H^1(\mathcal{F}^\vee \otimes \mathcal{O}_C(-1)) \otimes_k \mathcal{S} \xrightarrow{\text{id} \otimes \sigma} H^1(\mathcal{F}^\vee \otimes \mathcal{O}_C(-1)) \otimes_k \mathcal{O}_A^r$
- 3) $\text{Ext}_{\tilde{X} \times A/A}^1(\mathcal{G}_A, p^* \mathcal{F}) = H^1(\mathcal{F} \otimes \mathcal{O}_C(-m+1)) \otimes_k \mathcal{Q}^\vee \xrightarrow{\text{id} \otimes \tau^\vee} H^1(\mathcal{F} \otimes \mathcal{O}_C(-m+1)) \otimes_k (\mathcal{O}_A^r)^\vee$
- 4) $\text{Ext}_{\tilde{X} \times A/A}^1(\mathcal{G}_A, \mathcal{G}_A) = H^1(\mathcal{O}_C(-m)) \otimes_k \mathcal{H}om_A(\mathcal{Q}, \mathcal{S}) \xrightarrow{\text{id} \otimes (\tau^*, \sigma_*)} H^1(\mathcal{O}_C(-m)) \otimes_k \mathcal{E}nd_A(\mathcal{O}_A^r)$

where $0 \rightarrow \mathcal{S} \xrightarrow{\sigma} \mathcal{O}_A^r \xrightarrow{\tau} \mathcal{Q} \rightarrow 0$ is the canonical sequence on the Grassmannian A . In particular; if $\mathcal{F} = 0$ we get $k^{m-1} \times T_A^\vee \hookrightarrow k^{m-1} \times \mathcal{E}nd_A(\mathcal{O}_A^r)$, where $T_A^\vee = \mathcal{H}om_A(\mathcal{Q}, \mathcal{S})$ is the cotangent bundle on A .

Proof. The map is induced by $(\iota^{(-1)})^*$ and ι_* , see (11).

1. $\text{Ext}_{\tilde{X} \times A/A}^1(p^* \mathcal{F}, p^* \mathcal{F}) \cong R^1 q_* p^* \mathcal{E}nd_{\tilde{X}}(\mathcal{F}) \cong H^1(\mathcal{E}nd_{\tilde{X}}(\mathcal{F})) \otimes_k \mathcal{O}_A$ by (10). The restrictions of $\iota^{(-1)}$ and ι are the identity maps.

2. The restriction of $(\iota^{(-1)})^*$ is the identity while the restriction of ι_* is g_* . We shall obtain a commutative diagram

$$\begin{array}{ccc} R^1 q_* \mathcal{H}om_{\tilde{X} \times A}(p^* \mathcal{F}, \mathcal{G}_A) & \xrightarrow{g_*} & R^1 q_* \mathcal{H}om_{\tilde{X} \times A}(p^* \mathcal{F}, p^* \mathcal{O}_{\tilde{X}}(-D) \otimes q^* \mathcal{O}_A^r) \\ \alpha_1 \downarrow \cong & & \alpha_2 \downarrow \cong \\ H^1(\mathcal{F}^\vee \otimes \mathcal{O}_C(-1)) \otimes_k \mathcal{S} & \xrightarrow{\text{id} \otimes \sigma} & H^1(\mathcal{F}^\vee \otimes \mathcal{O}_C(-1)) \otimes_k \mathcal{O}_A^r \end{array}$$

Apply $\mathcal{H}om_{\tilde{X} \times A}(p^*\mathcal{F}, -)$ to (7) and then apply q_* to the resulting short exact sequence. We get a short exact sequence of R^1q_* -terms

$$0 \rightarrow R^1q_*\mathcal{H}om_{\tilde{X} \times A}(p^*\mathcal{F}, \mathcal{G}_A) \rightarrow$$

$$H^1(\tilde{X}, \mathcal{H}om_{\tilde{X}}(\mathcal{F}, \mathcal{O}_{\tilde{X}}(-D))) \otimes_k \mathcal{O}_A^r \xrightarrow{\eta^* \otimes \tau} H^1(\tilde{X}, \mathcal{H}om_{\tilde{X}}(\mathcal{F}, \mathcal{O}_C(-1))) \otimes_k \mathcal{Q} \rightarrow 0$$

since $H^i(\mathcal{F}^\vee(-C-D)) = 0$ for $i > 0$, and hence we also get the isomorphism α_1 . The α_2 is obtained analogously changing (7) to (4).

3. The restriction of ι_* is the identity map. The upper horizontal map in the following commutative diagram is by (10) the relevant inclusion.

$$\begin{array}{ccc} R^1q_*\mathcal{H}om_{\tilde{X} \times A}(\mathcal{G}_A, p^*\mathcal{F}) & \xrightarrow{(\iota^{(-1)})^*} & R^1q_*\mathcal{H}om_{\tilde{X} \times A}(p^*\mathcal{O}_{\tilde{X}}(-C-D) \otimes q^*\mathcal{O}_A^r, p^*\mathcal{F}) \\ \delta_1 \downarrow \cong & & \delta_2 \downarrow \cong \\ R^1q_*\mathcal{E}xt_{\tilde{X} \times A}^1(p^*\mathcal{O}_C(-1) \otimes q^*\mathcal{Q}, p^*\mathcal{F}) & \xrightarrow{\tau^*} & R^1q_*\mathcal{E}xt_{\tilde{X} \times A}^1(p^*\mathcal{O}_C(-1) \otimes q^*\mathcal{O}_A^r, p^*\mathcal{F}) \\ \delta_3 \uparrow \cong & & \delta_4 \uparrow \cong \\ H^1(\mathcal{F} \otimes \mathcal{O}_C(-m+1)) \otimes_k \mathcal{Q}^\vee & \xrightarrow{\text{id} \otimes \tau^\vee} & H^1(\mathcal{F} \otimes \mathcal{O}_C(-m+1)) \otimes_k (\mathcal{O}_A^r)^\vee \end{array}$$

The δ_1 is induced by the connecting map in the short exact sequence obtained by applying $\mathcal{H}om_{\tilde{X} \times A}(-, p^*\mathcal{F})$ to (7). But $R^iq_*(p^*\mathcal{F}(D)) \cong H^i(\mathcal{F}(D)) \otimes \mathcal{O}_A = 0$ for all $i > 0$ hence δ_1 is an isomorphism. Analogous arguing goes for δ_2 using the short exact sequence (4) instead. The δ_3 is the composition $H^1(\mathcal{F} \otimes \mathcal{O}_C(-m+1)) \otimes_k \mathcal{Q}^\vee \cong R^1q_*(p^*\mathcal{H}om_{\tilde{X} \times A}(\mathcal{O}_{\tilde{X}}(-C-D), \mathcal{F}) \otimes q^*\mathcal{Q}^\vee) \xrightarrow{\cong} R^1q_*\mathcal{E}xt_{\tilde{X} \times A}^1(p^*\mathcal{O}_C(-1) \otimes q^*\mathcal{Q}, p^*\mathcal{F})$ obtained by considering connecting maps induced from (4), analogous for δ_4 .

4. Substitute \mathcal{G}_A for $p^*\mathcal{F}$ in the previous diagram, analogous reasoning gives $(\iota^{(-1)})^*$ as

$$R^1q_*\mathcal{H}om_{\tilde{X} \times A}(p^*\mathcal{O}_{\tilde{X}}(C-D) \otimes q^*\mathcal{Q}, \mathcal{G}_A) \xrightarrow{\tau^*} R^1q_*\mathcal{H}om_{\tilde{X} \times A}(p^*\mathcal{O}_{\tilde{X}}(C-D) \otimes q^*\mathcal{O}_A^r, \mathcal{G}_A)$$

via the isomorphisms

$$\begin{aligned} R^1q_*\mathcal{E}nd_{\tilde{X} \times A}(\mathcal{G}_A) & \xrightarrow{\delta_1} R^1q_*\mathcal{E}xt_{\tilde{X} \times A}^1(p^*\mathcal{O}_C(-1) \otimes q^*\mathcal{Q}, \mathcal{G}_A) \\ & \xleftarrow{\delta_3} R^1q_*\mathcal{H}om_{\tilde{X} \times A}(p^*\mathcal{O}_{\tilde{X}}(C-D) \otimes q^*\mathcal{Q}, \mathcal{G}_A). \end{aligned}$$

essentially because $R^1q_*(p^*\mathcal{O}_{\tilde{X}}(D) \otimes \mathcal{G}_A) = 0$ which follows from (6). Apply

$$\mathcal{H}om_{\tilde{X} \times A}(p^*\mathcal{O}_{\tilde{X}}(-C-D) \otimes q^*\tau, -)$$

to (7) and obtain a map τ^* of short exact sequences. Applying q_* and the projection formula yields a map of short exact sequences in each cohomological degree essentially because $H^i(\mathcal{O}_{\tilde{X}}) = 0$ for $i > 0$. Calculating the two last R^1q_* -terms:

$$\begin{array}{ccc} H^1(\mathcal{O}_{\tilde{X}}(C)) \otimes_k \mathcal{H}om_A(\mathcal{Q}, \mathcal{O}_A^r) & \xrightarrow{\tau^*} & H^1(\mathcal{O}_{\tilde{X}}(C)) \otimes_k \mathcal{E}nd_A(\mathcal{O}_A^r) \\ H^1(\eta(C-D)) \otimes \tau_* \downarrow & & \downarrow H^1(\eta(C-D)) \otimes \tau_* \\ H^1(\mathcal{O}_C(-m)) \otimes_k \mathcal{E}nd_A(\mathcal{Q}) & \xrightarrow{\tau^*} & H^1(\mathcal{O}_C(-m)) \otimes_k \mathcal{H}om_A(\mathcal{O}_A^r, \mathcal{Q}) \end{array}$$

Again because $H^i(\mathcal{O}_{\tilde{X}}) = 0$ for $i > 0$, $H^1(\eta(C-D))$ is an isomorphism and the kernel of the vertical maps gives the relevant summand of $(\iota^{(-1)})^*$:

$$H^1(\mathcal{O}_C(-m)) \otimes_k \mathcal{H}om_A(\mathcal{Q}, \mathcal{S}) \xrightarrow{\tau^*} H^1(\mathcal{O}_C(-m)) \otimes_k \mathcal{H}om_A(\mathcal{O}_A^r, \mathcal{S}).$$

For ι_* , calculating

$$\mathcal{E}xt_{\tilde{X} \times A/A}^1(\mathcal{O}_{\tilde{X}}(-C-D) \boxtimes \mathcal{O}_A^r, \mathcal{O}_{\tilde{X}}(-D) \boxtimes \mathcal{O}_A^r) \cong H^1(\mathcal{O}_C(-m)) \otimes_k \mathcal{E}nd_A(\mathcal{O}_A^r)$$

gives the fourth summand of ι_* which is $g_* = \sigma_* : H^1(\mathcal{O}_C(-m)) \otimes_k \mathcal{H}om_A(\mathcal{O}_A^r, \mathcal{S}) \rightarrow H^1(\mathcal{O}_C(-m)) \otimes_k \mathcal{E}nd_A(\mathcal{O}_A^r)$. \square

Let $\text{Spec } k[\{w_{ij}\}_{1 \leq i \leq r}^{1 \leq j \leq s}] = \mathbb{A}_k^{sr}$ be the affine space $\text{Hom}_k(k^s, k^r)$ and let $U \subset \mathbb{A}^{sr}$ be the open sub-space of linear maps of maximal rank s . There is a principal fibre bundle $U \rightarrow A = \text{Grass}(s, r)$ with fibre $\text{Aut}_k(k^s)$. Let $h_1 = \dim_k H^1(\mathcal{E}nd \mathcal{F})$, $h_2 = \dim_k H^1(\mathcal{F}^\vee \otimes \mathcal{O}_C(-1))$, $h_3 = \dim_k H^1(\mathcal{F} \otimes \mathcal{O}_C(-m+1))$ and $h_4 = m-1$. Finally let

$$E = \text{Spec } \text{Sym}_k \text{Ext}_X^1(M, M)^* = \mathbb{A}_k^{h_1+r(h_2+h_3)+r^2(m-1)} = \text{Spec } k[\mathbf{u}, \mathbf{x}, \mathbf{y}, \mathbf{z}]$$

where $\mathbf{u} = \{u_i\}_{1 \leq i \leq h_1}$, $\mathbf{x} = \{x_i^{(n)}\}_{1 \leq i \leq r}^{1 \leq n \leq h_2}$, $\mathbf{y} = \{y_i^{(n)}\}_{1 \leq i \leq r}^{1 \leq n \leq h_3}$, $\mathbf{z} = \{z_{ij}^{(n)}\}_{1 \leq i, j \leq r}^{1 \leq n \leq m-1}$.

Corollary 1. Assume $s < r$. Let $\mathbb{T} \subset E \times U$ be the restriction of the closed affine scheme in $E \times \mathbb{A}_k^{sr}$ defined by the the following equations:

- i) The $s+1$ -minors of $((w_{ij})|x^{(n)})$ for $1 \leq n \leq h_2$
where $x^{(n)} = [x_1^{(n)}, \dots, x_r^{(n)}]^t$,
- ii) the entries of $[y_1^{(n)}, \dots, y_r^{(n)}](w_{ij})$ for $1 \leq n \leq h_3$,
- iii) the $s+1$ -minors of $((w_{ij})|z_j^{(n)})$ for $1 \leq j \leq r$ and $1 \leq n \leq m-1$
where $z_j^{(n)} = [z_{1j}^{(n)}, \dots, z_{rj}^{(n)}]^t$, and
- iv) the entries of $(z_{ij}^{(n)})(w_{ij})$ for $1 \leq n \leq m-1$.

Then the image R^d of T^d in E^h is the Henselisation of the image of \mathbb{T} under the projection $E \times U \rightarrow E$.

Remark 3. If $s = r$, $T^d = R^d = \mathbb{A}_k^{h_1+rh_2}$ which follows from Theorem 2.

Before proving Corollary 1 we give an application.

Definition 3. Suppose W is a k -vector space of dimension a , let

$$Y(a, b) = \{(y^{(1)}, \dots, y^{(b)}) \in W^b \mid \exists \text{ hyperplane } H \subset W \text{ with } y^{(i)} \in H \forall i\}$$

For $b < a$, $Y(a, b)$ is the whole affine space, but for $b \geq a$ it defines a closed cone in \mathbb{A}_k^{ab} .

Corollary 2. Assume $r > 1$. Let $C(P)$ be the affine cone over the image P of the Segre embedding $\mathbb{P}_k^{h_2+r(m-1)-1} \times \mathbb{P}_k^{r-1} \hookrightarrow \mathbb{P}_k^{(h_2+r(m-1))r-1}$. If $C(P)$, $Y(r, h_3)$ and $\mathbb{A}_k^{h_1}$ are embedded in E by the \mathbf{x} - and \mathbf{z} -coordinates, the \mathbf{y} -coordinates, and the \mathbf{u} -coordinates respectively, then the minimal stratum $R^d \subseteq E^h$, $d = c_1(\widetilde{M}) + C$, is given as the Henselisation of $C(P) \times Y(r, h_3) \times \mathbb{A}_k^{h_1} \subseteq E$ intersected with the locus defined by the equations

$$\begin{aligned} \text{tr}(z_{ij}^{(n)}) & \quad \text{with } 1 \leq n \leq m-1, \\ \sum_{i=1}^r x_i^{(n)} y_i^{(n')} & \quad \text{with } 1 \leq n \leq h_2, 1 \leq n' \leq h_3, \text{ and} \\ \sum_{i=1}^r y_i^{(n)} z_{ij}^{(n')} & \quad \text{with } 1 \leq n \leq h_3, 1 \leq n' \leq m-1, 1 \leq j \leq r. \end{aligned}$$

In particular; the minimal stratum is an isolated singularity if $M = \omega_X^{n_{m-2}} \oplus M_{m-1}^{n_{m-1}}$.

Remark 4. If $m = 2$, Corollary 2 and Theorem 2 strengthens [16, 5.6ii]. Corollary 2 also generalises and strengthens [16, 5.9].

Proof. Let $w = (w_{i1}) = [w_1, \dots, w_r]^t$. Equations i and iii in Corollary 1 say that the x -columns and the z -columns belong to the span of w . Hence there is a map $\mathbb{A}_k^{h_2+r(m-1)} \times (\mathbb{A}_k^r \setminus \{0\}) \rightarrow \mathbb{T}$ given by

$$([\alpha^{(1)}, \dots, \alpha^{(h_2)}], [\gamma^{(1)} | \dots | \gamma^{(m-1)}], w) \mapsto ([x^{(1)} | \dots | x^{(h_2)}], [(z_{ij}^{(1)}) | \dots | (z_{ij}^{(m-1)})], w)$$

where $x^{(n)} = \alpha^{(n)}w$ and $(z_{ij}^{(n)}) = w\gamma^{(n)}$ where $\gamma^{(n)} = [\gamma_1^{(n)}, \dots, \gamma_r^{(n)}]$, the image gives the locus defined by the equations *i* and *iii*. Projecting the image down to E (forgetting w) gives the Segre-cone. From *iv* we get $[w_i\gamma_1^{(n)}, \dots, w_i\gamma_r^{(n)}]w = 0$ for all $1 \leq i \leq r$ and since at least one w_i is invertible for each point in U we get $[\gamma_1^{(n)}, \dots, \gamma_r^{(n)}]w = 0$ or $\text{tr}(z_{ij}^{(n)}) = 0$. The $Y(r, h_r)$ -part is immediate from Definition 3 and *ii*. However; the w force relations between the y -, x - and z -variables. Equations *ii* say that the y -rows are orthogonal to w and hence to the x - and z -columns. Algebraically we eliminate the w_i from the equations $\text{rk}[w|x] \leq 1$ and $\sum y_i^{(n)}w_i$. E.g. if w_1 is invertible, $x_i = w_1^{-1}x_1w_i$ so multiplication by $w_1^{-1}x_1$ in $\sum y_iw_i$ gives $\sum x_iy_i$ for $x = x^{(n)}$ and $x = z_j^{(n)}$. For the last part of the statement, the module $\omega_X^{n_{m-2}} \oplus M_{m-1}^{n_{m-1}}$ has $h_1 = 0 = h_3$, and if $S^d := R^d \setminus \{0\}$ there is thus an inverse $S^d \rightarrow T^d \subseteq E^h \times \mathbb{P}^{r-1}$ to $T^d \rightarrow R^d$ where the second coordinate is given by the image of either a non-vanishing $(z_{ij}^{(n)})$ or a non-vanishing $x^{(n)}$. \square

Proof of Corollary 1. Pulling the canonical sequence on A back to U gives a short exact sequence of vector bundles $0 \rightarrow S \xrightarrow{(w_{ij})} V \otimes \mathcal{O}_U \rightarrow Q \rightarrow 0$ where $V \cong k^r$ and $S \cong \mathcal{O}_U^s$. Changing S to S , Q to Q and A with U in Theorem 2, the pulled back inclusion defines \mathbb{T} , and as the projection of \mathbb{T} to E factors through and surjects onto T^d , its image equals R^d . The $x_i^{(n)}$ -equations stem from inclusion 2 in Theorem 2 since the (free) algebra $\text{Sym}(S^\vee)$ is the quotient $\mathcal{O}_U[x_1, \dots, x_r]/\text{im } d_2$ where $\text{im } d_2$ is the ideal generated by the image of the second differential in the Buchsbaum-Rim complex of $(w_{ij})^t$, cf. [8, A2.10, A2.2], from which the equations are obtained. The $y_i^{(n)}$ -equations follows from $\text{Sym } Q \cong \mathcal{O}_U[y_1, \dots, y_r]/(\{\sum w_{ij}y_i\}_j)$, loc. cit. Finally, the natural map $V^\vee \otimes V \otimes \mathcal{O}_U \rightarrow S^\vee \otimes Q$ has kernel equal to the sum of the images of $Q^\vee \otimes V$ and $V^\vee \otimes S$. With $z_{ij}^{(n)}$ replacing $x_i^{(n)}y_j^{(n)}$, the $s+1$ -minors come from the image of $Q^\vee \otimes V$ while $(z_{ij}^{(n)})(w_{ij})$ comes from the image of $V^\vee \otimes S$, just as in the cases above. \square

Let the undecorated capital letters I, J and K in the following denote s -element sub-sets of $\{1, \dots, r\}$ and if $(u_{ij})_{1 \leq i \leq r, 1 \leq j \leq s}^{1 \leq j \leq s}$ is an $r \times s$ -matrix, let m_J^u denote the determinant of the sub-matrix $(u_{ij})_{i \in I}^{1 \leq j \leq s}$.

Corollary 3. *Let $I_d = (m_J^z)_{|J|=s}$ be the ideal generated by the s -minors of an arbitrary fixed set of s columns in $[(z_{ij}^{(1)}) | \dots | (z_{ij}^{(m-1)})]$ and let \tilde{R}^d be the strict transform of R^d under the blowing up of E^h in I_d . Then there is a natural embedding of \tilde{R}^d in $E^h \times A$ with image T^d .*

Proof. We show that the image of $\tilde{R}^d = \text{Proj}_{R^d}(\oplus I_d^i \cdot t^i)$ contains T^d and that they agree on an open set, hence since they both are irreducible they have to be equal. Let $N = \binom{r}{s} - 1$ and let $\{v_J\}_{|J|=s}$ be the global sections of $\mathcal{O}_{\mathbb{P}^N}(1)$, the Plücker embedding $A \hookrightarrow \mathbb{P}^N$ is defined by $v_J \mapsto m_J^w$, and the natural embedding $\tilde{R}^d \hookrightarrow R^d \times \mathbb{P}^N$ defined by $v_J \mapsto m_J^z \cdot t$ factorises through $R^d \times A$. We claim that the inverse image ideal sheaf $\mathcal{L} = I_d \cdot \mathcal{O}_{T^d}$ is an invertible sheaf globally generated by the minors in I_d considered as sections of \mathcal{L} . If $\mathcal{O}_{T^d}(1)$ defines the embedding $T^d \hookrightarrow E^h \times \mathbb{P}^N$, we want to show that the right triangle in the following diagram commutes.

$$\begin{array}{ccc}
 T^d & \xrightarrow{|\mathcal{O}(1)|} & E^h \times \mathbb{P}^{(r)-1} \\
 \downarrow & \searrow |\mathcal{L}| & \uparrow \text{nat.} \\
 R^d & \xleftarrow{\quad} & \tilde{R}^d
 \end{array}$$

This follows if there is an isomorphism $\mathcal{L} \cong \mathcal{O}_{T^d}(1)$ taking the global section $m_I^z \cdot t$ to m_I^w for all I . It is sufficient to show the identity $m_I^z m_J^w = m_I^w m_J^z$ in the function field of T^d for all I and J , and this will also prove our claim, cf. [13, 7.1, 7.14]. If $m_I^z m_J^w = m_I^w m_J^z$ and $m_J^z m_K^w = m_J^w m_K^z$ then $m_I^z m_K^w = m_I^w m_K^z$, hence it will be sufficient to show the identity for $I = \{1, 2, \dots, s\}$ and $J = \{2, 3, \dots, s+1\}$. Let z_j denote the j^{th} column of the selected s columns in $[(z_{ij}^{(1)}) | \dots | (z_{ij}^{(m-1)})]$; $z_j = [z_{1j}, \dots, z_{s+1,j}]^t$ and let W be the $(s+1) \times s$ -matrix $W = (w_{ij})_{1 \leq i \leq s+1}$. By expanding the relevant determinant in Corollary 1, equation (iii), along the last column, we get $\det[W|z_j] = z_{s+1,j} m_I^w - \sigma_j$ where $\sigma_j = \sum_{k=1}^s (-1)^{s-k} z_{kj} \tilde{m}_k$ and $\tilde{m}_k = m_I^w \{k\} \cup \{s+1\}$. Hence substituting $z_{s+1,j} = (m_I^w)^{-1} \sigma_j$ in m_J^z , we get $m_J^z = (m_I^w)^{-1} \left(\sum_{j=1}^s (-1)^{s+j} \sigma_j m_{\hat{j}} \right)$ where $m_{\hat{j}}$ is the $(s-1)$ -minor of $[z_1 | \dots | z_s]$ with the first row and the j^{th} column removed. Rearranging the sum, we have

$$\begin{aligned} m_J^z &= (m_I^w)^{-1} \left(\sum_{j=1}^s (-1)^{s+j} \sigma_j m_{\hat{j}} \right) = (m_I^w)^{-1} \left(\sum_{k=1}^s (-1)^{k+1} \tilde{m}_k \left(\sum_{j=1}^s (-1)^{j+1} z_{kj} m_{\hat{j}} \right) \right) \\ &= (m_I^w)^{-1} \tilde{m}_1 m_{\hat{1}} = (m_I^w)^{-1} m_J^w m_{\hat{1}}^z \end{aligned}$$

since $\sum_{j=1}^s (-1)^{j+1} z_{kj} m_{\hat{j}}$ is the determinant of (z_{ij}) where the first row is replaced by the k^{th} , which is zero except for $k = 1$.

Finally we note that restricted to the open set U_d in R^d where $[(z_{ij}^{(1)}) | \dots | (z_{ij}^{(m-1)})]$ has maximal rank, which is s , there is an inverse to the projection $T^d \twoheadrightarrow R^d$ which is defined by inserting the image of $[(z_{ij}^{(1)}) | \dots | (z_{ij}^{(m-1)})]$ as the second coordinate in $U_d \times A$. \square

Remark 5. The proof of Corollary 3 show that the $\text{rk} \leq s$ -locus in $\text{End}_k(V)$, $V \cong k^r$, which is defined by the $s+1$ -minors of the $r \times r$ -matrix (z_{ij}) , is resolved by blowing up \mathbb{A}^{r^2} in the ideal defined by the s -minors of any choice of s columns in (z_{ij}) . By the proof of Corollary 1 the resolving variety $\tilde{\mathbb{A}}^{r^2}$ is $V^\vee \times \mathcal{S}$. This shows that the filtration $\{R^d\}$ of the versal deformation space is closely related to the rank filtration of $\text{End}_k(V)$.

5. THE LOCAL DEFORMATION RELATION AND THE COMPONENTS

In Theorem 3 we count the components in the versal deformation space, and then we describe certain classes of modules which are interesting with respect to the local deformation relation.

Recall Definition 1 of the local deformation relation. A module will be called *terminal* if it does not locally deform to any other module and *initial* if no module locally deforms to it. It follows from [16, 4.9] (and the irreducibility of the T^d) that the terminal modules which M locally deforms to are in 1-to-1-correspondence with the geometric components in the versal deformation space of M as the fibre of the versal family corresponding to a generic point on the component. The *Chern number* is $c_1(\tilde{M}) \cdot C = \sum i \cdot n_i$. It uniquely determines the Chern class.

Theorem 3. *Let $r = \text{rk}(M) \geq 2$, $d = c_1(\tilde{M}) \cdot C$, $m \geq 3$, and suppose $\exists i \neq m-1$ such that $n_i > 0$. Then the terminal modules which M locally deforms to are given as $M_{i-1}^{n_{i-1}} \oplus M_i^{n_i}$ with $i = \lceil \frac{d'}{r} \rceil$, $n_{i-1} = ir - d'$, $n_i = r - n_{i-1}$ and $d' = d - jm$ for $\max(\lceil \frac{d+2r}{m} \rceil - r, 0) \leq j \leq \min(\lfloor \frac{d}{m} \rfloor, n_{m-1})$.*

In particular; the number of irreducible components in the reduced versal deformation space of Def_M is

$$1 + \min\left(\left\lfloor \frac{d}{m} \right\rfloor, n_{m-1}\right) - \max\left(\left\lceil \frac{d+2r}{m} \right\rceil - r, 0\right).$$

Remark 6. If $M = M_{m-1}^r$, $r \geq 2$, the only case (if $m \geq 3$) not covered by Theorem 3, then M deforms minimally to $M' = M_0 \oplus M_{m-2} \oplus M_{m-1}^{r-2}$ only (see Lemma 3), hence M and M' have the same terminal modules, and they have the same number of irreducible components in the versal deformation spaces.

The following key lemma describing the minimal local deformations and its first corollary will be useful in the proof of Theorem 3. The proof of Lemma 3 uses the geometrical description of R^d obtained in Theorem 1 to reduce the problem to a combinatorial one. Let (n_0, \dots, n_{m-1}) denote the reflexive module $M = M_0^{n_0} \oplus M_1^{n_1} \oplus \dots \oplus M_{m-1}^{n_{m-1}}$.

Lemma 3. *Suppose $m \geq 3$. The minimal Chern class preserving deformations are*

$$(n_0, \dots, n_j, 0, \dots, 0, n_i, \dots, n_{m-1}) \\ \dashrightarrow (n_0, \dots, n_{j-1}, n_j - 1, 1, 0, \dots, 0, 1, n_i - 1, n_{i+1}, \dots, n_{m-1})$$

for $3 \leq j+3 \leq i \leq m-1$, and

$$(n_0, \dots, n_{m-1}) \dashrightarrow (n_0, \dots, n_{i-2}, n_{i-1} - 1, n_i + 2, n_{i+1} - 1, n_{i+2}, \dots, n_{m-1})$$

for $1 \leq i \leq m-2$.

The minimal local deformations which change the Chern class (with $C \sim -mD$) are $(0, 0, \dots, 0, n_i, \dots, n_{m-1}) \dashrightarrow (1, 0, \dots, 0, 1, n_i - 1, n_{i+1}, \dots, n_{m-2}, n_{m-1} - 1)$ for $1 \leq i \leq m-2$ (if $i = 1$ then n_0 may be greater than 0 and n_0 increases by 2), and $(0, 0, \dots, 0, n_{m-1}) \dashrightarrow (1, 0, \dots, 0, 1, n_{m-1} - 2)$.

Proof. Assume M locally deforms to N and assume first that $c_1(\tilde{N}) = c_1(\tilde{M}) = d$. From [16, 4.9] $\text{Loc}(N) \subseteq R^d$, $A = \{\text{pt.}\}$ and $R^d = T^d$. From Theorem 1 $T^d = \text{Spec Sym}_k \text{Ext}_X^1(\tilde{M}, \tilde{M})^* \cong \mathbb{A}_k^N$. Consider a $z \in T(k)$ with $\mathcal{E}_z \cong \tilde{N}$ (\mathcal{E}_T as in the proof of Theorem 1), viewing z as a vector, it corresponds to an element ξ in the Zariski tangent space at the origin, i.e. $\xi \in \text{Ext}_X^1(\tilde{M}, \tilde{M})$. Consider the extensions

$$(14) \quad \eta: \quad 0 \longrightarrow \tilde{M}_j \xrightarrow{(u^{r-j}, v^{s-j})^t} \bigoplus_{\tilde{M}_s} \tilde{M}_r \xrightarrow{(v^{s-j}, -u^{r-j})} \tilde{M}_i \longrightarrow 0.$$

where $r + s = i + j$. (Remark that each of the modules in the middle have two non-isomorphic extensions producing them except if $r = s$.) If $M = M_j \oplus M_i$ with $j < i$ then

$$\text{Ext}_X^1(\tilde{M}, \tilde{M}) = \text{Ext}_X^1(\tilde{M}_i, \tilde{M}_j) \cong H^1(\mathcal{O}_{\tilde{X}}((j-i)D)) \cong H^1(\mathcal{O}_C(j-i)) \cong k^{i-j-1}$$

since $\text{Ext}_X^1(\tilde{M}_i, \tilde{M}_j) = 0$ if $j \geq i-1$. There is a local deformation from $M_j \oplus M_i$ to $M_r \oplus M_s$ defined by the $k[t]$ -flat family \mathcal{M}_t defined by the differential $d_{t,\eta} = \begin{pmatrix} d_j & t\eta \\ 0 & d_i \end{pmatrix}$ in a free resolution, i.e. $\mathcal{M}_0 = M_j \oplus M_i$ and $\mathcal{M}_t = M_r \oplus M_s$ if $t \in k^*$. Since $\text{Ext}_X^1(\tilde{M}, \tilde{M})$ is additive in both factors, a direct sum of a trivial extension and a η gives the elements in a k -basis for $\text{Ext}_X^1(\tilde{M}, \tilde{M})$. Writing ξ as a linear combination of η -s, there is a corresponding “stick figure” of lines connecting the origin and z by local deformations \mathcal{M}_t as above from M to N in R^d . Multiplying this stick figure by a scalar $t \in k^*$ does not alter the deformations, hence the \mathbb{A}^1 -family $\mathcal{E}_{t,z}$ gives N for $t \in k^*$. Remark the factorisation of local deformations $M_j \oplus M_i \dashrightarrow M_{j+1} \oplus M_{i-1} \dashrightarrow \dots \dashrightarrow M_r \oplus M_s$. It is left to the reader to show that the local deformations given in Lemma 3 are the only minimal ones among those induced by these.

If $d = c_1(\tilde{N}) = c_1(\tilde{M}) + sC$ for $s \geq 1$, by [16, 4.9] there exists a point $z \in T^d(k)$ mapping to a k -point in R^d with fibre N . In the vector bundle $T^d \rightarrow A$, the fibre over the base point $a \in A(k)$ of z is $T_a^d = \text{Spec Sym}_k \text{Ext}_X^1(\mathcal{E}, \mathcal{E})^*$ with its

universal extension $\mathcal{E}_{T_a} \cdot T_a^d$ maps isomorphically onto its image in R^d . We are essentially in the same situation as above, with \widetilde{M} replaced by \mathcal{E} . The only difference from the above argument is the additional k -basis elements in $\text{Ext}_{\widetilde{X}}^1(\widetilde{M}, \widetilde{M}^{\omega}) \cong \text{Ext}_X^1(M, M)$, see (12), given by extensions as in (14) where \widetilde{M}_j is replaced by $\mathcal{O}_{\widetilde{X}}(-D)$. \square

Remark 7. If $m = 2$ then $(n_0, n_1) \dashrightarrow (n_0 + 2, n_1 - 2)$ and the terminal modules are either $(r, 0)$ or $(r - 1, 1)$.

Definition 4. We refer to the directed graph representing the local deformation relation described in Lemma 3 as the *deformation graph*.

Corollary 4. *The initial modules with respect to the local deformation relation of rank r are $M_i \oplus M_{m-1}^{r-1}$ for $0 \leq i \leq m-1$, while the terminal modules are $M_{i-1}^{r-k} \oplus M_i^k$ for $1 \leq i \leq m-2$ and in addition M_{m-1} if $r = 1$.*

In particular; each connected component in the deformation graph has a unique initial node and for a given rank the graph has exactly m connected components, determined by the Chern number (mod m).

Proof. Applying Lemma 3 one checks that these indeed are the only initial (terminal) modules, for instance no initial module can have M_i^2 as a direct summand for $0 \leq i \leq m-2$. \square

Let us call M *central* if there are no modules with Chern class $c_1(\widetilde{M})$ locally deforming to M . Remark that for the rational double points all (reflexive) modules are central. The initial modules are clearly central. The central modules are obtained by adding free modules to the initial modules:

Corollary 5. *The central modules of rank r are $M_0^k \oplus N$ where N is initial of rank $r - k$ for all $0 \leq k \leq r$. The local deformation relation induces a total ordering of the central modules of rank r and Chern number $j \pmod{m}$.*

In particular; to each M there is a unique central module with Chern class $c_1(\widetilde{M})$ locally deforming to M .

The proof is similar to the proof of Corollary 4. Observe that the initial module $M_i \oplus M_{m-1}^{r-1}$ locally deforms to the *centrally terminal* module $M_0^{r-1} \oplus M_j$ where $j \equiv i - r + 1 \pmod{m}$ and that the centrally terminal modules are exactly the central modules with Chern number less than m . Also remark that the central modules are determined by the Chern character, i.e. $(\text{rk}(M), c_1(\widetilde{M}))$.

Corollary 6. *If M is initial of rank $r \geq 2$ and $m \geq 3$ then the number of irreducible components in the reduced versal deformation space of Def_M is:*

$$1 + r - 2\left\lceil \frac{r}{m} \right\rceil \quad \text{if } M = M_{m-1}^r \text{ with } r \geq 2, \text{ and}$$

$$\begin{cases} 1 + r - \left\lceil \frac{r+i+1}{m} \right\rceil & \text{if } r \leq i+1 \\ 1 + r - \left\lceil \frac{r+i+1}{m} \right\rceil - \left\lceil \frac{r-i-1}{m} \right\rceil & \text{if } r > i+1 \end{cases} \quad \text{and } M = M_i \oplus M_{m-1}^{r-1}$$

for $0 \leq i \leq m-2$. In particular; the number of irreducible components in the reduced versal deformation space of any reflexive module N of rank $r \geq 2$ on X is less than or equal to $1 + r - \left\lceil \frac{2r}{m} \right\rceil$.

Proof. For $M = M_{m-1}^r$ apply $(0, 0, \dots, 0, r) \dashrightarrow (1, 0, \dots, 0, 1, r-2)$, then use Theorem 3. \square

Proof of Theorem 3. The idea is first to show that we have a unique terminal module with the given invariants r and d , if it exists. By Corollary 4 the terminal module is of the form $M_{i-1}^{n_{i-1}} \oplus M_i^{n_i}$ for $1 \leq i \leq m-2$ and the numerical conditions give $n_{i-1} + n_i = r$ and $(i-1)n_{i-1} + in_i = d$. We would like to find i . Since $ir - d + n_i = r$ and $1 \leq n_i \leq r$, i is uniquely determined by the inequalities

$0 \leq ir - d \leq r - 1$. Such an i gives us a module if and only if $i = \lceil \frac{d}{r} \rceil \leq m - 1$. We also have $n_{i-1} = ir - d$ and $n_i = r - n_{i-1}$. However; this module is terminal if and only if $i \leq m - 2$, i.e. if and only if $\frac{d}{r} \leq m - 2$. Otherwise there is no terminal module with the given invariants and we have to apply a Chern class changing deformation to get further. By Lemma 3 this can be done by subtracting 1 from n_{m-1} by assumption. Remark that the new module can not be M_{m-1}^r since it is initial by Corollary 4. We may try to solve the equations for i again, with d changed to $d - m$ and so on. If $\frac{d}{r} > m - 2$, let l be the smallest number of Chern class changing deformations which have to be applied to get $\frac{d-lm}{r} \leq m - 2$, then $l = \lceil \frac{d+2r}{m} \rceil - r$. In any case the first Chern number d_{\max} for which there is a terminal module is $d_{\max} = d - \max(l, 0)m$. We may lower the Chern number by jm (in j steps) if $d - jm \geq 0$ and $n_{m-1} - j \geq 0$, hence the last Chern number d_{\min} for which there is a component is $d_{\min} = d - \min(\lfloor \frac{d}{m} \rfloor, n_{m-1})m$. \square

From Theorem 3 (and Remark 6) we immediately get the following corollary:

Corollary 7. *If M is reflexive and not free on X and $d = c_1(\widetilde{M}) \cdot C$ then M locally deforms to a free module if and only if $d \equiv 0 \pmod{m}$ and $n_{m-1} \geq \frac{d}{m}$.*

Corollary 8. *The multiplicity of M_{m-1} in M is an upper semi-continuous invariant. The k -dimension of $\text{Ext}_{\widetilde{X}}^1(\widetilde{M}, \widetilde{M})$ is strictly decreasing for Chern class preserving local deformations.*

Proof. From [16, 4.10] it follows that $c_1(\widetilde{M}) \cdot C$ is an upper semi-continuous invariant and the Chern number decreases if and only if the multiplicity of M_{m-1} decreases by Lemma 3. For the second statement, the change of $\text{Ext}_{\widetilde{X}}^1(\widetilde{M}, \widetilde{M})$ -dimension for the two minimal Chern class preserving local deformations is $-(n_i + n_j) \leq -2$ and $-(n_{i-1} + n_{i+1}) + 1 \leq -1$ respectively. \square

Remark 8. The following example shows that $\dim_k \text{Ext}_{\widetilde{X}}^1(\widetilde{M}, \widetilde{M})$ is insufficient to separate reflexive modules with the same rank and Chern class. We draw the part of the deformation graph with constant Chern number 6 beginning with $M = \mathcal{O}_X \oplus M_2 \oplus M_4$:

$$(1, 0, 1, 0, 1) \begin{matrix} \searrow \\ \swarrow \end{matrix} \begin{matrix} (0, 2, 0, 0, 1) \\ (1, 0, 0, 2, 0) \end{matrix} \begin{matrix} \searrow \\ \swarrow \end{matrix} \begin{matrix} (0, 1, 1, 1, 0) \\ (0, 0, 3, 0, 0) \end{matrix}$$

We have $\dim_k \text{Ext}_{\widetilde{X}}^1 = 4$ for both $(0, 2, 0, 0, 1)$ and $(1, 0, 0, 2, 0)$.

6. EXAMPLES

Example 1. Let $m = 3$ and $M = M_1 \oplus M_2^2$, then $c_1(\widetilde{M}) \cdot C = 5$. We consider T^2 and the image R^2 in $E = \mathbb{A}^{10}$ ($h_2 = 1$, $h_1 = 0 = h_3$) gives the closure of the minimal stratum which in this case is the whole reduced versal deformation space R_{red} . Since $s = 1$ and $r = 2$, $A = \text{Grass}(1, 2) = \mathbb{P}^1$. We have $\mathcal{S} = \mathcal{O}_{\mathbb{P}^1}(-1) = \mathcal{Q}^\vee$ and from Theorem 2 we get that

$$T^2 = \mathbf{Spec} \text{Sym}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)).$$

and $\dim R = 4$. From Corollary 2, R_{red} and therefore (in the present situation) R is an isolated singularity. We see that it contains an affine plane and 2-dimensional cones of multiplicity m , with $2 \leq m \leq 4$ (we have surjections $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(m)$ for $1 \leq m \leq 4$), and $T^2 \rightarrow R_{\text{red}}$ is a small resolution with a \mathbb{P}^1 as exceptional divisor.

Example 2. We consider another simple example. Let $m = 3$ and $M = M_2^3$. The deformation graph is:

$$\begin{array}{ccc} (0, 0, 3) & & d = 6 \\ \downarrow & & \\ (1, 1, 1) & \dashrightarrow & (0, 3, 0) \\ \downarrow & & d = 3 \\ (3, 0, 0) & & d = 0 \end{array}$$

From this we see that there are two components corresponding to the two terminal modules; M_1^3 with Chern number 3 and \mathcal{O}_X^3 with Chern number 0. We have $c_1(\widetilde{M}) \cdot C = 6$ and T^6 is just a point since $s = 0$ so $A = \text{Grass}(0, 3)$ and $\text{Ext}_X^1(\widetilde{M}, \widetilde{M}) = 0$. By Theorem 2 we have

$$T^3 = \mathbf{Spec} \text{Sym}_{\mathcal{O}_{\mathbb{P}^2}}(T_{\mathbb{P}^2} \oplus T_{\mathbb{P}^2})$$

and $\dim R^3 = 6$ since $s = 1$ and $A = \text{Grass}(1, 3)$, and $E = \text{Spec Sym}_k \text{Ext}_X^1(M, M)^* = \text{Spec } k[\{z_{ij}^{(n)}\}] = M_{3,3} \times M_{3,3}$ where $M_{3,3}$ denotes the affine space of 3×3 -matrices. By Corollary 1 the embedding of $T^3 \subseteq E \times \mathbb{P}^2$ is given by the minors of

$$\begin{bmatrix} w_1 & z_{1j}^{(1)} \\ w_2 & z_{2j}^{(1)} \\ w_3 & z_{3j}^{(1)} \end{bmatrix} \quad \text{and of} \quad \begin{bmatrix} w_1 & z_{1j}^{(2)} \\ w_2 & z_{2j}^{(2)} \\ w_3 & z_{3j}^{(2)} \end{bmatrix} \quad \text{for } 1 \leq j \leq 3$$

and the entries of

$$\begin{bmatrix} z_{11}^{(1)} & z_{12}^{(1)} & z_{13}^{(1)} \\ z_{21}^{(1)} & z_{22}^{(1)} & z_{23}^{(1)} \\ z_{31}^{(1)} & z_{32}^{(1)} & z_{33}^{(1)} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad \text{and of} \quad \begin{bmatrix} z_{11}^{(2)} & z_{12}^{(2)} & z_{13}^{(2)} \\ z_{21}^{(2)} & z_{22}^{(2)} & z_{23}^{(2)} \\ z_{31}^{(2)} & z_{32}^{(2)} & z_{33}^{(2)} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

The image of the projection of T^3 to E is R^3 and by Corollary 2 R^3 is given as the cone over the image of the Segre embedding $\mathbb{P}^5 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}(M_{3 \times 3} \times M_{3 \times 3})$ given by

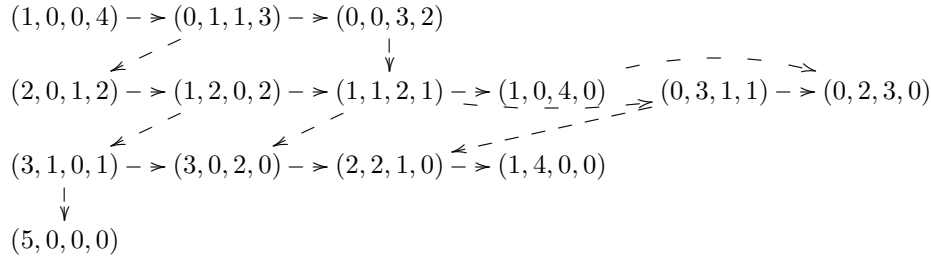
$$(15) \quad \left(\left(\begin{bmatrix} \gamma_1^{(1)} \\ \gamma_2^{(1)} \\ \gamma_3^{(1)} \end{bmatrix}, \begin{bmatrix} \gamma_1^{(2)} \\ \gamma_2^{(2)} \\ \gamma_3^{(2)} \end{bmatrix} \right), \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right) \mapsto \left(\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \begin{bmatrix} \gamma_1^{(1)} \\ \gamma_2^{(1)} \\ \gamma_3^{(1)} \end{bmatrix}^t, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \begin{bmatrix} \gamma_1^{(2)} \\ \gamma_2^{(2)} \\ \gamma_3^{(2)} \end{bmatrix}^t \right)$$

intersected with the two hyperplanes $z_{11}^{(n)} + z_{22}^{(n)} + z_{33}^{(n)}$, $n = 1, 2$.

By Theorem 2 we get $T^0 \cong T^3$, hence $R^0 \cong R^3$. But since T^0 is a bundle over the dual projective plane $\text{Grass}(2, 3) = \check{\mathbb{P}}^2$, the correct embedding of $T^0 \hookrightarrow E \times \check{\mathbb{P}}^2$ is given by the map in (15) with the $(z_{ij}^{(n)})$ -matrices transposed. It follows that the intersection $R^3 \cap R^0$ is given by the image under (15) of the locus where the gamma vectors are parallel, which thus is 5-dimensional of codimension one in R . On the intersection the fibre of the versal family must be $N = \mathcal{O}_X \oplus M_1 \oplus M_2$ as N is the only reflexive module besides M which locally deforms to both the terminal modules.

Example 3. If $m = 4$ there are 4 connected components in the deformation graph which together contain all reflexive modules on X of rank 5. The components have unique initial modules, one of them is $M = \mathcal{O}_X \oplus M_3^{\oplus 4}$ which determines the

following deformation graph component:



The Chern numbers of the rows are 12, 8, 4 and 0 respectively. We see that the reduced versal deformation space has three components corresponding to the terminal modules $(0, 2, 3, 0)$, $(1, 4, 0, 0)$ and $(5, 0, 0, 0)$. The intersection of the components is non-trivial, and at a general point in the intersection the fibre of the versal family is $(1, 2, 0, 2)$ and one can argue that the dimension of the intersection is at least 7.

Finally, if $s = 2$, then $A = \text{Grass}(2, 4)$ and $T^4 = T_{M_R}^4$ is by Theorem 2 given as

$$T^4 = \mathbf{Spec} \, \text{Sym}_{\mathcal{O}_A}(\mathcal{Q}^2 \oplus T_A^3).$$

and $\dim R^4 = 20$. At a general point in the component the fibre in the versal family is the terminal module $\mathcal{O}_X \oplus M_1^4$ and the map $T^4 \rightarrow R^4$ gives a resolution of the component with central fibre $\text{Grass}(2, 4)$. This resolution is obtained by blowing up the ideal generated by the 2-minors of a 4×2 -matrix given by any two columns from the \mathbf{z} -matrices, see Corollary 3.

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